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Fano colourings of cubic graphs and the Fulkerson Conjecture

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Abstract

A Fano colouring is a colouring of the edges of a cubic graph by points of the Fano plane such that the colours of any three mutually adjacent edges form a line of the Fano plane. It has recently been shown by Holroyd and Škoviera [Colouring of cubic graphs by Steiner triple systems, J. Combin. Theory Ser. B 91 (2004) 57–66] that a cubic graph has a Fano colouring if and only if it is bridgeless. In this paper we prove that six, and conjecture that four, lines of the Fano plane are sufficient to colour any bridgeless cubic graph. We establish connections of our conjecture to other conjectures concerning bridgeless cubic graphs, in particular to the well-known conjecture of Fulkerson about the existence of a double covering by 1-factors in every bridgeless cubic graph.

Keywords: Cubic graph; Edge-colouring; Fano plane; Snark; Fulkerson Conjecture

1. Introduction

A *Fano colouring* is an edge-colouring of a cubic graph which uses points of the Fano plane as colours subject to the condition that any three colours meeting at a vertex form a line (Figs 1 and 2). With the classical concept of a Tait colouring Fano colourings share the property that the colours of any two adjacent edges determine the colour of the third edge adjacent to them. Moreover, a colouring which uses the same line at all vertices is nothing but the usual 3-edge-colouring. This makes Fano colourings a natural generalization of Tait colourings and a suitable tool for investigating cubic graphs that are not 3-edge-colourable—that is, *snarks*.

It is convenient to consider Fano colourings within a broader context of edge-colourings by Steiner triple systems. Recall that a *Steiner triple system* $\mathcal{S} = (X, B)$ of order n is a collection B of three-element subsets (called *triples* or *blocks*) of a set X of n *points* such that each pair of points is together present in exactly one triple. Given a Steiner triple system $\mathcal{S}$, an $\mathcal{S}$-*colouring* of a cubic graph G is a colouring of the edges of G by points of $\mathcal{S}$ such that the three colours occurring at any vertex form a block of $\mathcal{S}$. (We allow our graphs to have multiple edges and sometimes even loops. However, edge-colourings make sense only for loopless graphs.)

The study of edge-colourings by Steiner triple systems was initiated by Archdeacon [1], and the first result in this direction is due to Fu [8] who described two classes of bridgeless cubic graphs that admit a Fano colouring. Recently,

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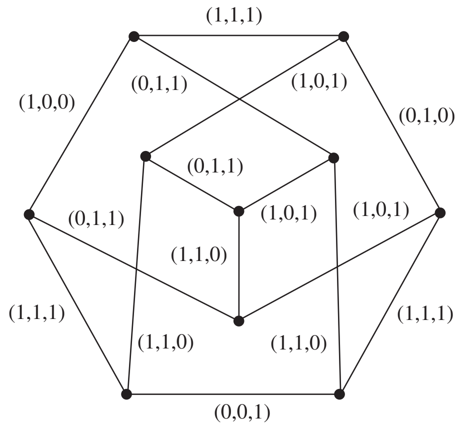

Fig. 1. A Fano colouring of the Petersen graph.

Holroyd and Škoviera [12] substantially improved Fu's results by showing that every bridgeless cubic graph has an $\mathcal{S}$-colouring for every Steiner triple system $\mathcal{S}$ of order greater than 3.

This paper is devoted to an investigation of colourings which employ the smallest non-trivial Steiner triple system, the Fano plane. As follows from [12], every bridgeless cubic graph has a Fano colouring. Here we will deal with further properties of Fano colourings. Our main concern is the following problem: *How many lines of the Fano plane are needed to colour a given cubic graph*?

At the first glance, one could expect that the answer to this question might involve the structure of lines employed by a Fano colouring. Surprisingly, it only depends on their number. In particular, any Fano colouring of a non-3-edge-colourable graph requires at least four lines, and there is only one admissible configuration of four lines which can occur. On the other hand, all seven lines are never needed.

Theorem 1.1. *Every bridgeless cubic graph has a Fano colouring which uses at most six different lines.*

These facts suggest that all snarks fall into one of three classes according to the number of lines required by a Fano colouring. While the class of graphs that require four lines is infinite (see Theorem 4.1 and Example 4.2), we have not been able to find any representatives of the remaining two classes. Moreover, with the help of a computer we have verified that up to 30 vertices no snarks of this sort exist. This justifies the following conjecture.

Conjecture 1.2. *Every bridgeless cubic graph has a Fano colouring which uses at most four lines.*

It was conjectured by Fulkerson [9] that in every bridgeless cubic graph there is a collection of six perfect matchings such that each edge belongs to exactly two of them. We show that Fulkerson's conjecture implies Conjecture 1.2, and that the latter conjecture is equivalent to the statement that every bridgeless cubic graph contains three perfect matchings with empty intersection, conjectured by Fan and Raspaud in [7]. Finally, we propose two weaker versions of Conjecture 1.2 involving configurations of five lines.

2. Colourings and configurations

We start with a definition of the Fano plane (Fig. 2). Here is one of the possibilities: The *Fano plane* is an incidence structure $\mathcal{F} = (P, L)$ consisting of a set P of seven *points*, say $P = \{1, 2, \ldots, 7\}$, and a collection of seven *lines* $L = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 7\}, \{3, 5, 6\}\}$. A point p and a line l such that $p \in l$ are said to be *incident*.

If we label each point $i \in P$ by its binary code $(i_1, i_2, i_3) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, we obtain the usual representation of $\mathcal{F}$ as the projective plane $PG(2, 2)$ of order 7. Hence, the following three axioms are satisfied in $\mathcal{F}$:

(P1) There is exactly one line through any pair of distinct points.