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Geometric Hermite Interpolation—In memoriam Josef Hoschek

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Abstract

In this paper we present an overview over the more recent developments of Geometric Hermite Approximation Theory for planar curves. A general method to solve those problems is presented. Emphasis is put on the relations to differential geometry and to invariance against parameter transformations and the motion group of the underlying geometry. However, besides a few elementary cases, this leads to nonlinear systems of algebraic equations.

Furthermore we give some geometric interpretations, a couple of examples and a detailed discussion of the case degree $n = 4$ with one contact point.

Keywords: Hermite interpolation; Geometric splines; Approximation order

1. The general problem of Geometric Hermite Interpolation

1.1. Introduction

Geometric Hermite Interpolation (GHI) is a wide generalization of different approximation schemes that have been developed in the past for parametric curves to be approximated by polynomial or rational curves. The underlying basic concept is to consider the curve itself *independently of its actual parametrization*.

Commonly, a curve[1] will not be approximated by a single polynomial or rational curve, but by a *geometric spline*, i.e., a finite number of consecutive segments, joining at the breakpoints with an appropriate

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[1] In practice, only compact curves are considered.

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order of geometric continuity. This latter notion—not being so clear[2] in the early period of CAGD—is defined as follows:

Definition 1. Two parametric curves

$$\mathbf{x}: I \to \mathbb{R}^d, \qquad \mathbf{y}: I^* \to \mathbb{R}^d$$

(with $I = [t_0, t_1] \subset \mathbb{R}$, $I^* = [s_0, s_1] \subset \mathbb{R}$, $t_0 < t_1$, $s_0 < s_1$) are said to "join with geometric continuity of order k" (or shortly "… with G^k-continuity") iff the following conditions hold:

(a) The mappings $\mathbf{x}, \mathbf{y}$ are of differentiability class C^k in some extended open sets $S, S^* \subset \mathbb{R}$ containing I and I^*, respectively.
(b) The end point of $\mathbf{x}$ coincides with the starting point of $\mathbf{y}$:

$$\mathbf{x}(t_1) = \mathbf{p} = \mathbf{y}(s_0).$$

(c) There is a diffeomorphism[3] $\phi: U \to V$ mapping a neighbourhood U of t_1 monotonously increasing onto a neighbourhood V of s_0 ($U \subset S \subset \mathbb{R}$, $V \subset S^* \subset \mathbb{R}$) such that $\phi(t_1) = s_0$.
(d) At this common point, the derivatives up to the order k coincide in the following sense ("after reparametrization by ϕ"):

$$\left.\frac{\mathrm{d}^j \mathbf{x}(t)}{\mathrm{d}t^j}\right|_{t=t_1} = \left.\frac{\mathrm{d}^j \mathbf{y}(\phi(t))}{\mathrm{d}t^j}\right|_{t=t_1} \qquad (j = 0, \ldots, k). \tag{1}$$

Clearly, G^k-continuity implies $G^{k'}$-continuity for all $k' < k$, and G^0-continuity is equivalent to (b). For G^1-continuity, the coincidence of the oriented tangents at $\mathbf{p}$ (in addition to (a)) is sufficient. For planar curves, G^2-continuity holds if, additionally, the two curvatures are the same; in spatial or higher dimensional cases, the osculating planes must coincide, too. But it must be observed, that G^2 and coincidence of the torsions is *not sufficient* for G^3-continuity! (see (Degen, 1988)).

In cases where the two curves share an *interior* point one says "they have a contact of order k" at that common point.

Applying the (iterated) chain rule to (1) leads to the conditions

$$\begin{aligned}
&\mathbf{x}' = \phi' \dot{\mathbf{y}},\\
&\mathbf{x}'' = \phi'' \dot{\mathbf{y}} + (\phi')^2 \ddot{\mathbf{y}},\\
&\mathbf{x}''' = \phi''' \dot{\mathbf{y}} + 3\phi'\phi'' \ddot{\mathbf{y}} + (\phi')^3 \dddot{\mathbf{y}},\\
&\ldots
\end{aligned} \tag{2}$$

(written up to the order three) where primes denote derivation wrt t and dots those wrt s. (All arguments are omitted; put t_1 for $\mathbf{x}^{(k)}$ and $\phi^{(k)}$ and s_0 for $\mathbf{y}^{(k)}$. Later on we write ϕ_k instead of $\phi^{(k)}$; see Section 1.2.)

For higher orders, these formulas become more and more complicated. But their principal structure is that the derivatives of $\mathbf{x}$ are expressed by the product of a matrix T with those of $\mathbf{y}$ up to the same order; the matrix T is a lower left triangular matrix with the powers of ϕ' as diagonal elements, hence a *regular* matrix. It is called the "connection matrix" or "the adjacence matrix".

[2] Other terms like "visual continuity" oder "Frenet frame continuity" (even not being equivalent!) appeared.
[3] I.e., a bijective C^∞-function with C^∞ inverse.