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New bounds on the magnitude of the derivative of rational Bézier curves and surfaces

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Abstract

Two bounds on the magnitude of the derivative of rational Bézier curves are presented and compared with the known ones. As an application, we improve the existing bounds on the magnitude of partial derivatives of rational Bézier surfaces.

Keywords: Rational Bézier curve; Rational Bézier surface; Derivative; Bound

1. Introduction

A rational Bézier curve γ of degree n is given by the control points $P_i \in \mathbf{R}^d$ and positive weights $w_i \in \mathbf{R}$ in the Form

$$\gamma(t) = \frac{\sum_{i=0}^{n} B_i^n(t) w_i P_i}{\sum_{i=0}^{n} B_i^n(t) w_i}, \quad 0 \leqslant t \leqslant 1, \tag{1.1}$$

where B_i^n are the Bernstein polynomials given by $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$.

Several authors gave bounds on the magnitude of the derivative, see (Floater, 1992; Hermann, 1999; Zhongke et al., 2004). The last article develops a straightforward way to compute a bound by expressing the derivative in rational Bézier form and exploiting the convex hull property. Hermann considered degrees two and three. The results of the current work partially improve Floater's inequalities

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$$\|\gamma'\| \leqslant n\frac{\max_i w_i}{\min_i w_i}\cdot\max_{i,j}\|P_i - P_j\|, \tag{1.2}$$

$$\|\gamma'\| \leqslant n\left(\frac{\max_i w_i}{\min_i w_i}\right)^2\cdot\max_{i}\|P_{i+1} - P_i\|. \tag{1.3}$$

Here and in the sequel max and min are formed over all possible indices. It is already stated in (Floater, 1992) that neither bound is stronger than the other.

We derive the two following new inequalities:

$$\|\gamma'\| \leqslant n\max\left\{\omega, \frac{1}{\omega}\right\}\cdot\max_{i,j}\|P_i - P_j\|, \tag{1.4}$$

$$\|\gamma'\| \leqslant n\max\left\{\omega, \frac{1}{\omega}\right\}^n\cdot\max_{i}\|P_{i+1} - P_i\|, \tag{1.5}$$

where $\omega := \max_i \frac{w_i}{w_{i+1}}$. (1.4) is an improvement of (1.2), while (1.5) gives a better bound than (1.3) in some cases.

The inequalities will be applied to surfaces in the last section, by which we get an improvement of the results in (Wang et al., 1997).

2. Curves

We follow the approach of Floater (1992), and exploit the representation of the derivative:

$$\gamma'(t) = n\frac{w_0^{n-1}w_1^{n-1}}{(w_0^n)^2}\left(P_1^{n-1} - P_0^{n-1}\right). \tag{2.1}$$

Here w_i^k and P_i^k are the intermediate weights and points of the de Casteljau algorithm. By setting $w_i^0 = w_i$ and $P_i^0 = P_i$, they are given by:

$$w_i^k = (1-t)w_i^{k-1} + tw_{i+1}^{k-1}, \tag{2.2}$$

$$w_i^k P_i^k = (1-t)w_i^{k-1}P_i^{k-1} + tw_{i+1}^{k-1}P_{i+1}^{k-1}. \tag{2.3}$$

First we show the following lemma.

Lemma 2.1. *For the intermediate weights w_i^k of the kth step of the de Casteljau algorithm, we have*:

$$\frac{w_i^k}{w_{i+1}^k} \leqslant \max_j \frac{w_j^{k-1}}{w_{j+1}^{k-1}} \quad \textit{and} \quad \frac{w_{i+1}^k}{w_i^k} \leqslant \max_j \frac{w_{j+1}^{k-1}}{w_j^{k-1}}.$$

Proof. For positive real numbers a, b, c, d and $t \in [0, 1]$, we have:

$$\min\left\{\frac{a}{c}, \frac{b}{d}\right\} \leqslant \frac{(1-t)a + tb}{(1-t)c + td} \leqslant \max\left\{\frac{a}{c}, \frac{b}{d}\right\}.$$

Application to

$$\frac{w_i^k}{w_{i+1}^k} = \frac{(1-t)w_i^{k-1} + tw_{i+1}^{k-1}}{(1-t)w_{i+1}^{k-1} + tw_{i+2}^{k-1}}$$

completes the proof. □