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Agnostic G^1 Gregory surfaces

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ABSTRACT

We discuss G^1 smoothness conditions for rectangular and triangular Gregory patches. We then incorporate these G^1 conditions into a surface fitting algorithm. Knowledge of the patch type is inconsequential to the formulation of the G^1 conditions, hence the term *agnostic G^1 Gregory surfaces*.

1. Introduction

Surfaces are used for many modeling purposes, ranging from car bodies or airplane fuselages to objects in animated movies or interactive games. Depending on the application at hand, different surface types are used, such as spline surfaces [1] for the first two examples and subdivision surfaces [2] for the last two.

Spline surfaces cover a model with rectangular patches, which can create problems in areas where triangular shapes are needed. Subdivision surfaces have potential problems because direct evaluation is possibly slow [3]. For this reason, several authors have studied polynomial or rational polynomial approximation subdivision surfaces [4–6].

In this paper, we investigate spline-like surfaces which cover a model by a mix of triangular and rectangular patches. These are rational polynomial patches, first investigated by Gregory [7] in rectangular form and by Walton and Meek [8] in triangular form. Our surfaces are G^1, meaning they have continuous tangent planes everywhere. This is in contrast to spline surfaces, which are typically second order differentiable, or C^2.

First we introduce rectangular and triangular Gregory surfaces. Next we introduce our G^1 conditions. We then incorporate these G^1 conditions into a surface fitting algorithm.

2. Rectangular Gregory surfaces

A bicubic Bézier patch is given by a control net

$$\begin{matrix} \mathbf{b}_{00} & \mathbf{b}_{01} & \mathbf{b}_{02} & \mathbf{b}_{03} \\ \mathbf{b}_{10} & \mathbf{b}_{11} & \mathbf{b}_{12} & \mathbf{b}_{13} \\ \mathbf{b}_{20} & \mathbf{b}_{21} & \mathbf{b}_{22} & \mathbf{b}_{23} \\ \mathbf{b}_{30} & \mathbf{b}_{31} & \mathbf{b}_{32} & \mathbf{b}_{33} \end{matrix}$$

and, for a point $\mathbf{b}(u, v)$ on the patch:

$$\mathbf{b}(u, v) = \sum_{i=0}^{3}\sum_{j=0}^{3}\mathbf{b}_{ij}\frac{36}{i!j!(3-i)!(3-j)!}(1-u)^{3-i}u^i(1-v)^{3-j}v^j,$$

where the parametric domain is given by $0 \leqslant u,\ v \leqslant 1$. The 3D points $\mathbf{b}_{ij}$ form a control net which determines the shape of the patch.

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A "bicubic"[1] Gregory patch [9] is given by a control net of the same structure but with variable interior control points

$$\mathbf{b}_{11} = \frac{u}{u+v}\mathbf{b}_{11}^{10} + \frac{v}{u+v}\mathbf{b}_{11}^{01},$$
$$\mathbf{b}_{21} = \frac{1-u}{1-u+v}\mathbf{b}_{21}^{10} + \frac{v}{1-u+v}\mathbf{b}_{21}^{01},$$
$$\mathbf{b}_{12} = \frac{u}{1-v+u}\mathbf{b}_{12}^{10} + \frac{1-v}{1-v+u}\mathbf{b}_{12}^{01},$$
$$\mathbf{b}_{22} = \frac{1-u}{2-u-v}\mathbf{b}_{22}^{10} + \frac{1-v}{2-u-v}\mathbf{b}_{22}^{01}.$$

The superscript 10 identifies Gregory control points with greater influence on the boundaries, where u varies, and likewise, the superscript 01 identifies Gregory points with more influence on the boundaries, where v varies. Fig. 1 illustrates a bicubic Gregory patch.

The eight interior control points might come from cross boundary continuity conditions. In that context, we will be interested in the degree 3×1 surface formed by the two rows of control points along each edge, called the *tangent ribbon*. Thus the tangent ribbon defines the tangent plane along the boundary. The ribbons along $v = 0$ and $v = 1$ are given by control points

$$\begin{matrix} \mathbf{b}_{00} & \mathbf{b}_{01} \\ \mathbf{b}_{10} & \mathbf{b}_{11}^{10} \\ \mathbf{b}_{20} & \mathbf{b}_{21}^{10} \\ \mathbf{b}_{30} & \mathbf{b}_{31} \end{matrix} \quad \text{and} \quad \begin{matrix} \mathbf{b}_{02} & \mathbf{b}_{03} \\ \mathbf{b}_{12}^{10} & \mathbf{b}_{13} \\ \mathbf{b}_{22}^{10} & \mathbf{b}_{23} \\ \mathbf{b}_{32} & \mathbf{b}_{33}, \end{matrix}$$

respectively. The ribbons along $u = 0$ and $u = 1$ are given by control points

$$\begin{matrix} \mathbf{b}_{00} & \mathbf{b}_{01} & \mathbf{b}_{02} & \mathbf{b}_{03} \\ \mathbf{b}_{10} & \mathbf{b}_{11}^{01} & \mathbf{b}_{12}^{01} & \mathbf{b}_{13} \end{matrix} \quad \text{and} \quad \begin{matrix} \mathbf{b}_{20} & \mathbf{b}_{21}^{01} & \mathbf{b}_{22}^{01} & \mathbf{b}_{23} \\ \mathbf{b}_{30} & \mathbf{b}_{31} & \mathbf{b}_{32} & \mathbf{b}_{33} \end{matrix}$$

respectively. Fig. 2 (left) illustrates a tangent ribbon for a bicubic Bézier patch.

3. Triangular Gregory surfaces

A quartic triangular Bézier patch is given by the control net

$$\begin{matrix} & & & & \mathbf{b}_{040} \\ & & & \mathbf{b}_{031} & & \mathbf{b}_{130} \\ & & \mathbf{b}_{022} & & \mathbf{b}_{121} & & \mathbf{b}_{220} \\ & \mathbf{b}_{013} & & \mathbf{b}_{112} & & \mathbf{b}_{211} & & \mathbf{b}_{310} \\ \mathbf{b}_{004} & & \mathbf{b}_{103} & & \mathbf{b}_{202} & & \mathbf{b}_{301} & & \mathbf{b}_{400} \end{matrix}$$

and, for a point $\mathbf{b}(u, v, w)$ on the patch:

$$\mathbf{b}(u, v, w) = \sum_{i+j+k=4} \frac{24}{i!j!k!} u^i v^j w^k \mathbf{b}_{ijk},$$

where the parametric domain is given by barycentric coordinates $u + v + w = 1$.

A triangular Gregory patch [8] is given by a control net of the same structure but with variable interior control points

[1] The so-called bicubic Gregory patch is rational and degree seven in both u and v.

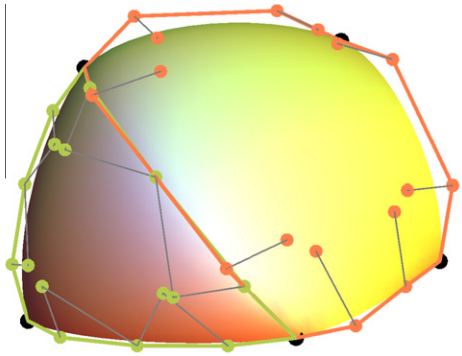

Fig. 1. Orange control points: A bicubic rectangular Gregory patch. Green control points: a quartic triangular Gregory patch. Control points are connected to the boundaries to which they yield more influence. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$$\mathbf{b}_{112} = \frac{u}{u+v}\mathbf{b}_{112}^{101} + \frac{v}{u+v}\mathbf{b}_{112}^{011},$$
$$\mathbf{b}_{211} = \frac{w}{w+v}\mathbf{b}_{211}^{101} + \frac{v}{w+v}\mathbf{b}_{211}^{110},$$
$$\mathbf{b}_{121} = \frac{u}{u+w}\mathbf{b}_{121}^{110} + \frac{w}{u+w}\mathbf{b}_{121}^{011}.$$

The superscript 101 identifies a Gregory control point with more influence on the $(u,0,w)$ boundary, the superscript 110 identifies a Gregory point with more influence on the $(u,v,0)$ boundary, and the superscript 011 identifies a Gregory point with more influence on the $(0,v,w)$ boundary. Fig. 1 illustrates a quartic triangular Gregory patch.

Here we will use a special quartic patch in which the three quartic boundary curves are degree elevated cubics. (This point will be revisited in Sections 4 and 5.) Let the cubic representation of these boundary curves be as follows:

$$\begin{matrix} & & & \mathbf{c}_{030} \\ & & \mathbf{c}_{021} & & \mathbf{c}_{120} \\ & \mathbf{c}_{012} & & & & \mathbf{c}_{210} \\ \mathbf{c}_{003} & & \mathbf{c}_{102} & & \mathbf{c}_{201} & & \mathbf{c}_{300} \end{matrix}$$

Now the tangent ribbons are defined as follows. The ribbon along $u = 0$ is given by control points

$$\begin{matrix} & & & \mathbf{c}_{030} & \mathbf{c}_{120} \\ & & \mathbf{c}_{021} & \mathbf{b}_{121}^{011} \\ & \mathbf{c}_{012} & \mathbf{b}_{112}^{011} \\ \mathbf{c}_{003} & \mathbf{c}_{102} \end{matrix}$$

The ribbon along $v = 0$ is given by control points

$$\begin{matrix} \mathbf{c}_{012} & \mathbf{b}_{112}^{101} & \mathbf{b}_{211}^{101} & \mathbf{c}_{210} \\ \mathbf{c}_{003} & \mathbf{c}_{102} & \mathbf{c}_{201} & \mathbf{c}_{300}. \end{matrix}$$

The ribbon along $w = 0$ is given by control points

$$\begin{matrix} \mathbf{c}_{021} & \mathbf{c}_{030} \\ & \mathbf{b}_{121}^{110} & \mathbf{c}_{210} \\ & & \mathbf{b}_{211}^{110} & \mathbf{c}_{120} \\ & & & \mathbf{c}_{201} & \mathbf{c}_{300}. \end{matrix}$$