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Free-form splines combining NURBS and basic shapes ☆

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ABSTRACT

We show how to automatically join, into one unified spline surface, C^2 tensor-product bi-cubic NURBS and G^2 bi-cubic rational splines. The G^2 splines are capable of exactly representing basic shapes such as (pieces of) quadrics and surfaces of revolution, including tori and cyclides. The main challenge is to transition between the differing forms of continuity. We transform the G^2 splines to splines that are C^2 in homogeneous space. This yields Hermite data for a transitional strip of tensor-product splines of degree (6,5) that guarantees overall curvature continuity. We also explain the simpler G^1 to C^1 transition. Key to the constructions is the C^2 parameterization of circles in homogeneous space.

1. Motivation

Applications, such as ball and socket joints, require designs to locally exactly reproduce parts of basic shapes, such as spheres, or more generally quadrics and rotational objects. Inclusion of such basic shapes can also serve to define and, for their extent within the design surface, guarantee fair surface shape. At the same time, downstream processing of the resulting design surfaces favors representation of basic shapes and C^2 splines in one standard form, namely as NURBS of moderate degree. Many downstream problems can be avoided if the combined NURBS surface, of basic shapes and free-form tensor-product splines, is one consistent structure without the piecemeal stitching by separate blends and fillets.

This challenge has motivated us to take a fresh look at both the theory of rational geometric splines and of reproducing conics, constructions based on circles in particular [9,8,7]. Rational geometric splines have been developed as early as [2,5] – but this classical work focused on algebraic generality and fails to provide constructive recipes. In particular, it does not address the challenge of setting the many scalar degrees of freedom of geometric continuity and rational weights that determine the quality of the resulting curves and surfaces. On the other hand, the piecemeal reproduction of conic segments by rational pieces in Bernstein-Bézier form (see e.g. [4,10]) lacks the built-in smoothness required of a unified structure with unique control points. The trilogy [9,8,7] provides building blocks for reproducing multiple basic shapes and automatically joining them smoothly, both for curves and surfaces. It offers G^1 continuous constructions of degree bi-2, and G^2 continuity of degree bi-3; the scalars defining geometric continuity and rational weights are algorithmically initialized to reproduce the basic shapes.

The present paper adds to this framework an automatic and structurally efficient curvature-continuous transition (the gray strip in Fig. 1b) between standard C^2 bi-3 tensor-product free-form splines (green[1]) and the basic shape

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[1] For interpretation of color in Figs. 1, 2, 5, 7, 9–15, the reader is referred to the web version of this article.

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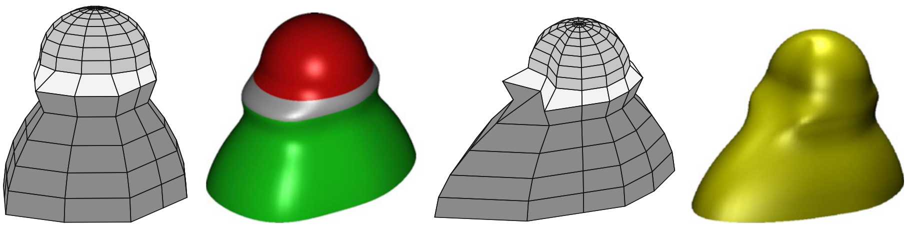

Fig. 1. Sphere to NURBS transition. (a) The light gray control structure defines a spherical dome, the dark gray structure a C^2 spline patch. (b) The combined surface joins the sphere-topped dome (red) and the spline (green) via a transition strip (gray) into a C^2 NURBS surface. (c) The corresponding NURBS mesh can be freely modified as a rational spline resulting in the surface (d).

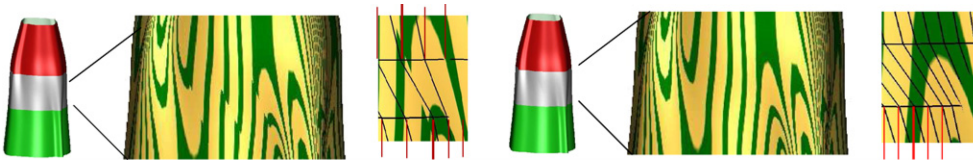

Fig. 2. Failure of naive Hermite interpolation. (a) Discontinuous reflection lines when directly interpolating the existing spline surfaces (red, green). The enlargement (right-most) shows additionally the control net (black) and illustrates the non-matching β_i, top vs. bottom, by the vertical red line segment spacing; thicker lines indicate patch boundaries. (b) Smooth reflection lines of our unified representation.

(red). Note that the corresponding unified control net, behaves again like a NURBS control net (cf. Fig. 1c and d). The key challenge addressed is that free-form splines and rational basic surfaces typically have different parameters of geometric continuity – in the direction parallel to the interface. This means that their bi-3 pieces cannot be merged directly without damage to the surface quality. (In [8], the parameters of the splines could be chosen to match those of the basic surfaces.) Fig. 2a illustrates the negative effect on surface quality of naive second-order Hermite interpolation across the boundaries. This motivates our more elaborate approach, whose core will be seen to be an extension of two approaches to C^2 circle parameterization [3,9]. We use them to convert G^2 to C^2 joins and then automatically generate a transition strip from the resulting Hermite data as in Fig. 2b.

1.1. Outline

In Section 2, we review the constructions that allow basic shapes to be part of a modifiable curvature-continuous rational bi-3 spline representation; and we explain why we use a parameterization of degree 6 that is controlled via a degree 3 spline. In Section 3, we introduce the two alternative reparameterizations for turning, in Section 4, G^2 splines of degree bi-3 into C^2 splines of degree 6. Section 4 explains how to join these degree 6 splines with standard bi-3 NURBS, to form a unified spline surface with high-quality transitions. Section 4 also provides several examples of unified C^2 spline surfaces. In Section 5, we briefly sketch the much simpler unification of rational G^1 splines with bi-2 C^1 NURBS. Explicit formulas for the reparameterizations are listed in the Appendix.

2. Basic shapes in homogeneous rational form, circles and C^2 splines of degree 6

2.1. Basic shapes in homogeneous rational form

Section 3.1 of Karčiauskas and Peters [8] gives a recipe for representing a wide variety of basic shapes, including quadrics and generalized tori such as cyclides, as rational bi-cubic splines such that the pieces join with curvature continuity (G^2, see below). Viewed as maps $\mathbb{R}^2 \to \mathbb{R}^4$, i.e. in homogeneous form, these exact representations of the basic shapes as splines have coordinate functions that are expressed, piecewise, in rational Bernstein-Bézier form (cf. [4,10]):

$$f(u,v) := \frac{\sum_{r=0}^{3}\sum_{s=0}^{3} w_{rs} b_{rs} B_r(u) B_s(v)}{\sum_{r=0}^{3}\sum_{s=0}^{3} w_{rs} B_r(u) B_s(v)}.$$

$$B_k(t) := \binom{3}{k}(1-t)^{3-k} t^k. \tag{1}$$

The bi-cubic construction in [8] provides explicit formulas for exactly representing the basic shapes by a control structure consisting of points in $\mathbb{R}^3$, weights and scalars of geometric continuity. This control structure can be freely manipulated, by moving its control points, changing weights or scalars, while automatically retaining smoothness. This enables a design work-flow that starts with the designer selecting basic shapes, an automatic algorithmic conversion to the mentioned control structure that the designer can freely manipulate, possibly via higher-order tools, and finally an automatic conversion of the modified control structure to a curvature continuous surface.