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Creases and boundary conditions for subdivision curves

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ABSTRACT

Our goal is to find subdivision rules at creases in arbitrary degree subdivision for piecewise polynomial curves, but without introducing new control points e.g. by knot insertion. Crease rules are well understood for low degree (cubic and lower) curves. We compare three main approaches: knot insertion, ghost points, and modifying subdivision rules. While knot insertion and ghost points work for arbitrary degrees for B-splines, these methods introduce unnecessary (ghost) control points.

The situation is not so simple in modifying subdivision rules. Based on subdivision and subspace selection matrices, a novel approach to finding boundary and sharp subdivision rules that generalises to any degree is presented. Our approach leads to new higher-degree polynomial subdivision schemes with crease control without introducing new control points.

1. Introduction

We wish to have arbitrary-degree subdivision surfaces with creases and boundary conditions that are as robust as those available for B-spline surfaces [8]. The existing methods for degree 3 do not generalise to higher degrees. We have analysed the problem and provide a general solution, with specific worked examples for degrees up to 7. We present here our results for curves, which provide the necessary precursor to the more challenging surface cases.

Sharp creases and end-point interpolation (including Bézier end-conditions) in B-spline curves (and by extension in tensor-product B-spline surfaces) are typically achieved via multiple knots. Indeed, a knot of multiplicity m reduces the continuity of a degree d B-spline to C^{d-m} from the native C^{d-1} continuity at single knots. Thus, to create a crease, a knot of multiplicity d can be used. To achieve Bézier end-conditions, knots of multiplicity $d+1$ are included at the start and end of knot vectors.

Instead of using multiple knots to achieve end-point interpolation, one can use ghost (also known as phantom or virtual) points. Depending on degree, several ghost points are pre- and appended to the control polygon. These are carefully placed (as linear combinations of existing control points) so that the resulting curve satisfies given end-conditions. This technique yields modified basis functions formed as linear combinations of B-splines.

A popular alternative to using basis functions to evaluate spline curves and surfaces is recursive subdivision [18,20]. Creases and boundary interpolation rules can still be obtained via multiple knots [17,16,12], but there is an alternative available: smooth subdivision rules are modified to sharp ones [9,7]. This has the advantage over multiple knots that the user does not need to interact with the knot vector. The user marks control vertices of a curve (edges in the surface case) as smooth (default) or sharp. This leads to an intuitive modelling interface as no extra control points are introduced, in contrast to knot insertion.

Motivated by these observations and the fact that sharp rules have so far been limited to low-degree subdivision [7,1,17,15,16,11,10], we investigate a more general setting

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for introducing sharp creases and boundary interpolation rules in higher-degree spline curves. Our results then extend naturally to tensor-product surfaces and potentially to higher-degree subdivision surfaces, such as those by Stam [24] and Cashman [2].

The problem of finding crease rules, and our approach to solving it, comprise our main contribution (Section 3). We present case studies for odd degrees (Section 4) and the more challenging even degrees (Section 5), demonstrated on examples of B-spline subdivision curves with creases. We show that relaxing some of our requirements (Section 6) leads to interesting trade-offs between the simplicity of subdivision rules and the behaviour of subdivision curves at creases and end-points. Before all this, we present our notation and a summary of the necessary underlying B-spline theory.

2. Preliminaries

Consider a polynomial spline curve of degree d and order $k = d + 1$ given by the knot vector $\mathbf{t} = (t_0, t_1, \ldots, t_{n+d}), t_i \leqslant t_{i+1}$, where $i = 0, \ldots, n + d - 1$, and by n control points $\mathbf{P}_i$:

$$\mathbf{c}(t) = \sum_{i=0}^{n-1} B_{i,k}(t)\mathbf{P}_i, \quad t_{k-1} \leqslant t \leqslant t_n. \tag{1}$$

The B-splines $B_{i,k}$ are defined recursively [6]:

$$B_{i,1}(t) = \begin{cases} 1 & \text{if } t_i \leqslant t < t_{i+1}, \\ 0 & \text{otherwise}, \end{cases}$$
$$B_{i,k} = \frac{t - t_i}{t_{i+k-1} - t_i} B_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} B_{i+1,k-1}(t), \tag{2}$$

with the convention $\frac{0}{0} = 0$. It is typically required that $t_i < t_{i+d}$ for all $i = 1, \ldots, n - 1$. From this definition it follows that the support of $B_{i,k}$, i.e., the closure of the interval where it is non-zero, is $[t_i, t_{i+k}]$.

While many of the ideas that we explore below can be applied in the general setting of non-uniform knot vectors, we focus on initially uniform knot vectors $t_i = i$, but knots are subsequently allowed to become multiple. An example of uniform B-splines is shown in Fig. 1a. To achieve Bézier end-conditions, an open-uniform knot vector (end knots have multiplicity k) can be used; see Fig. 1b.

2.1. B-spline creases

The typical B-spline approach to creating sharp creases is by using multiple knots. This follows from the fact that the continuity of a B-spline of degree d at a knot of multiplicity m is C^{d-m}. In the cubic case, a triple knot is used. For an existing curve, there are two variants. First, one moves two knots to create three coalescing knots, i.e., a triple knot; see Fig. 1c. Second, one inserts a desired knot several times until its multiplicity reaches $m = d$; see Fig. 1d. This introduces new control points that the user can freely move around. While valid and popular, these solutions are not ideal, especially when generalised to tensor-product surfaces, for the following reasons:

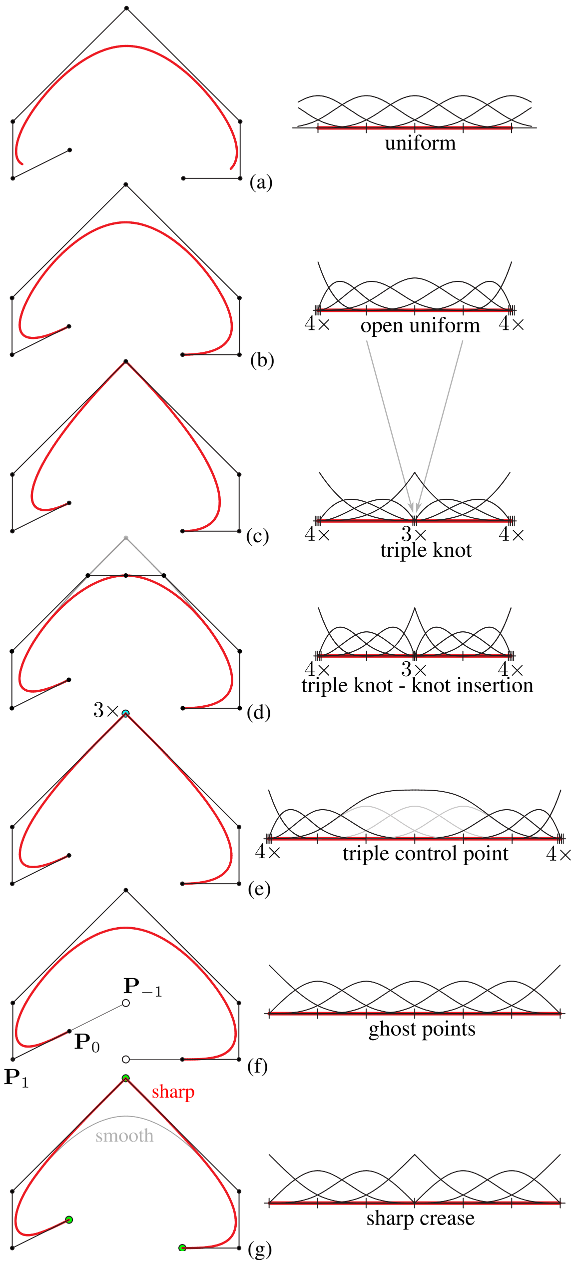

Fig. 1. A comparison of various cubic splines (left) and the basis functions (right) used to generate them. All basis functions are either cubic B-splines or obtained as their linear combinations. (a–d) The flexibility offered by modelling systems that allow the user to modify knot vectors. Note that (c) results from (d) by moving two knots to create a knot of multiplicity 3. (e) The effect of a triple control point (cyan) and the corresponding basis. (f) Ghost points can be used to force end-point interpolation without modifying the knot vector. (g) Modifying subdivision rules to allow for control points to be tagged either as smooth (default) or sharp (green) offers intuitive control over the resulting spline. End-points are marked as sharp implicitly even for smooth curves (shown in grey) that have no internal points marked as sharp. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

- the user needs to have access to the knot vector and understand how creating and moving multiple knots influences the shape of a curve or surface;