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Curvature-based blending of closed planar curves

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ABSTRACT

A common way of blending between two planar curves is to linearly interpolate their signed curvature functions and to reconstruct the intermediate curve from the interpolated curvature values. But if both input curves are closed, this strategy can lead to open intermediate curves. We present a new algorithm for solving this problem, which finds the closed curve whose curvature is closest to the interpolated values. Our method relies on the definition of a suitable metric for measuring the distance between two planar curves and an appropriate discretization of the signed curvature functions.

1. Introduction

Shape blending or *shape morphing* is a very active research field in computer graphics, which deals with the mathematical theory and the algorithms for constructing a gradual and continuous transformation between two planar or solid shapes. The problem is typically divided into two steps: the *vertex correspondence problem*, which establishes a correspondence between the two shapes [1–5], and the *vertex path problem*, which actually determines the interpolated shape. This paper focusses on the second step, and while most existing shape blending methods deal with polygonal shapes [6–8], we consider the case where source and target shape are smooth.

Given two planar parametric curves $\gamma_0 : I_0 \to \mathbb{R}^2$ and $\gamma_1 : I_1 \to \mathbb{R}^2$, the problem of blending between these two curves is to find for any $t \in [0,1]$ a curve $\gamma_t : I_t \to \mathbb{R}^2$ such that the mapping $t \mapsto \gamma_t$ is at least continuous in t and reproduces γ_0 for $t = 0$ and γ_1 for $t = 1$. One can then interpolate continuously between γ_0 and γ_1 by varying the parameter t.

If we assume that γ_0 and γ_1 are parameterized over a common interval $I = I_1 = I_2$, then the simplest blending is given by $\gamma_t : I \to \mathbb{R}^2$, $\gamma_t(s) = (1-t)\gamma_0(s) + t\gamma_1(s)$, but this choice is undesirable for two reasons. First, it depends on the particular parameterizations of γ_0 and γ_1 and second, it can lead either to naturally (see Fig. 1, first row) or to unnaturally looking intermediate curves (see Fig. 2, first row).

A more promising approach [9] is to define the intermediate curves by linearly interpolating the signed curvature functions of γ_0 and γ_1 and to reconstruct the intermediate curve γ_t from the interpolated curvature values. However, if γ_0 and γ_1 are closed, then this procedure can result in an open curve γ_t, which is again undesirable. Surazhsky and Elber [9] fix this problem by adapting the strategy of Sederberg et al. [10] to close γ_t in a post-processing step. Note that the idea of working in *curvature space* is also useful for computing isometric curvature flow [11].

1.1. Contributions

In this paper we propose an alternative algorithm for finding a blending between two closed curves which

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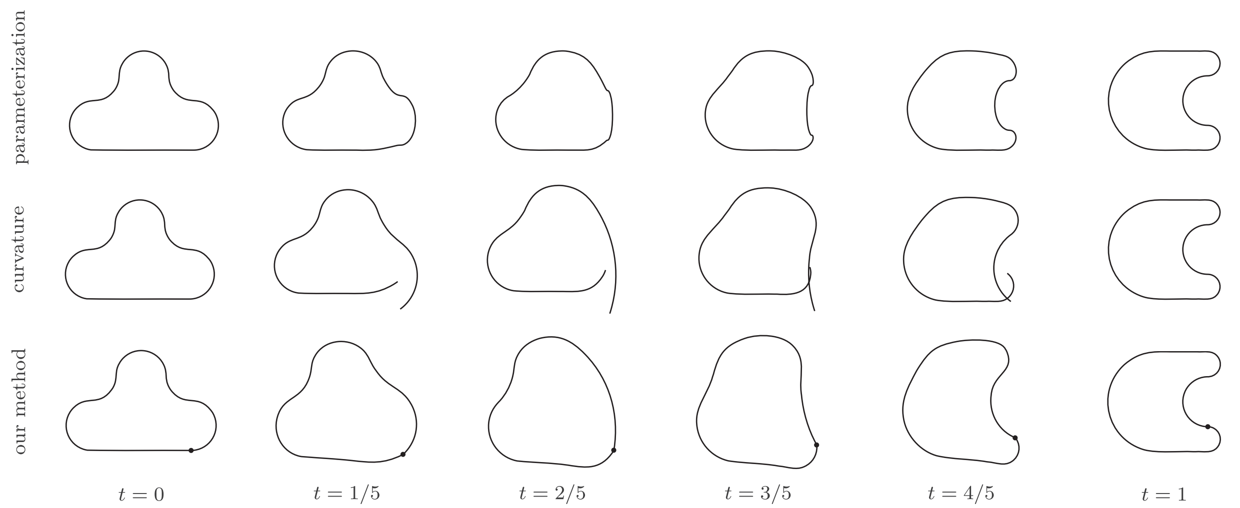

Fig. 1. Deformation of the source curve γ_0 (left) into the target curve γ_1 (right). The top row shows the parameterization-based interpolant, the middle row the curvature-based interpolant, and the third row the results of our closing process. The dot indicates the start and end point of the curves.

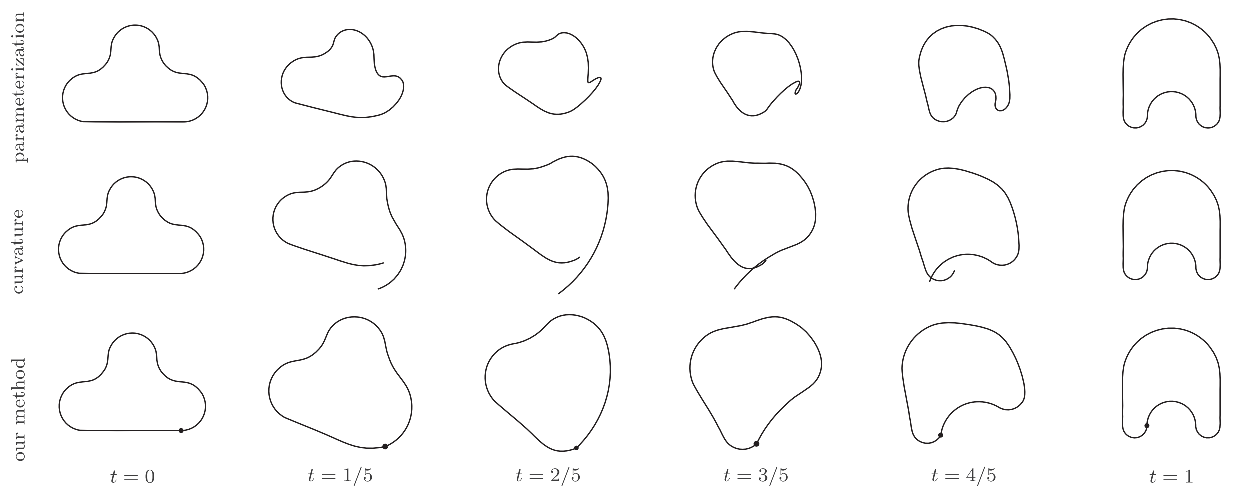

Fig. 2. Deformation of the same input curves as in Fig. 1, except that the target curve γ_1 is rotated clockwise by ninety degrees. The top row shows the parameterization-based interpolant, the middle row the curvature-based interpolant, and the third row the results of our closing process. Note that curvature-based interpolation is invariant with respect to the orientations of the input shapes, while parameterization-based interpolation depends on it.

guarantees that the intermediate curves are closed, too. We first provide a precise mathematical statement of the problem and its theoretical solution (Section 2) and then introduce an adequate discretization of the signed curvature function of a given curve (Section 3). We finally use this idea to come up with a practical solution (Section 4) and report a number of examples and comparisons to the state of the art (Section 5).

2. Theoretical background

Let the two given curves γ_0 and γ_1 be C^2-continuous[1] and closed, and assume without loss of generality that they are parameterized with respect to *arc length* over the intervals $I_0 = [0, L_0]$ and $I_1 = [0, L_1]$, respectively, where L_i is the *length* of the curve γ_i. That is, $\gamma_i(0) = \gamma_i(L_i)$ and $\|\gamma_i'(s)\| = 1$ for any $s \in I_i$. The signed curvature of γ_i at $\gamma_i(s)$ is then given by $k_i(s) = \det(\gamma_i'(s), \gamma_i''(s))$ and we call $k_i : I_i \to \mathbb{R}$ the *signed curvature function* of γ_i.

It is well known [12] that for any given signed curvature function $k : [0, L] \to \mathbb{R}$ there exists a *unique* regular planar curve $\gamma : [0, L] \to \mathbb{R}^2$ up to rigid motions, such that the signed curvature function of γ with respect to arc length is exactly k, and there even exists an explicit way of constructing γ. We simply let

$$\theta_s = \int_0^s k(u)du + \theta_0, \tag{1a}$$

and then

[1] We actually allow for the slightly weaker assumption that both curves are C^1 and piecewise C^2 with bounded curvature.

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