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Hierarchical B-splines on regular triangular partitions

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ABSTRACT

Multivariate splines have a wide range of applications in function approximation, finite element analysis and geometric modeling. They have been extensively studied in the last several decades, and specially the theory on bivariate B-splines over regular triangular partition is well developed. However, the above mentioned splines do not have local refinement property – a property that is very important in adaptive function approximation and level of detailed representation of geometric models. In this paper, we introduce the concept of hierarchial bivariate splines over regular triangular partitions and construct basis functions of such spline space that satisfy some nice properties. We provide some examples of hierarchial splines over triangular partitions in surface fitting and in solving numerical PDEs, and the results turn out to be promising.

1. Introduction

In CAD/CAM industry, free form surfaces are usually represented by tensor product polynomials or rational maps, such as tensor-product B-splines and NURBS. For these standard tensor-product representations, local adaptive refinement is naturally supported by the hierarchical spline model, where different levels of details are identified by means of a hierarchy of tensor-product splines [1]. Based on such hierarchical model, complex surfaces can be created from simple NURBS surfaces with hierarchical editing.

Compared to tensor-product representations, splines on triangular partition have the advantage of flexibility and lower degree with the same continuity. The theory on multivariate splines over triangular partitions has been widely studied in the past several decades, see [2] for a detailed survey. However, for arbitrary triangular partition, the spline space depends on not only the topology but also the geometry of the partition, leading to high cost on computation and difficulties in controlling the spline functions. Thus, instead of working with splines over arbitrary triangulation, we focus our attention on bivariate splines over two regular triangular partitions, called type-I triangular partition and type-II triangular partition respectively. The theory of bivariate splines defined on type-I and type-II triangular partition has been well developed by Renhong Wang et al. [2]. Unfortunately, such kind of splines does not have local refinement property, which limits its applications in Computer Aided Design (CAD) and finite element analysis.

In this paper, we extend the hierarchical representations from tensor-product splines to splines defined on regular triangular partitions. The idea is to define a set of basis functions by certain rules from a sequence of nested bivariate spline spaces defined on a nested regular triangular partition. The refinement domain can be any type which is convenient to capture local details. We call such hierarchial model on regular triangular partition as hierarchical bivariate splines. The functions in such spline space have some nice properties which are useful in finite element analysis and CAD.

2. Related work

To address the problem of local refinement of splines on rectangular domain, the concept of hierarchical B-splines

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(HB-splines for short) was firstly introduced by Forsey and Bartels as an accumulation of B-splines with nested knot vectors [1]. HB-splines can be locally refined using overlays. Later researches mainly focus on how to construct bases of hierarchical B-spline spaces. The first specific basis selection mechanism was proposed by Kraft in [3], and extended by Vuong et al. in [4]. The basis functions of the hierarchical spline space constructed in [4] are non-negative, linearly independent and locally supported. Shortly after, Jüttler et al. normalized the hierarchical B-splines proposed in [4] by reducing the support of basis functions defined on coarse grids, according to finer levels in the hierarchy of splines [5]. They call such hierarchical B-splines as truncated hierarchical B-splines (THB-splines), which are non-negative, locally supported, linearly independent and form a partition of unity, and allow an effective local control of refinement.

There are also several other kinds of local refinement splines developed in the past decade besides HB-splines. *T-splines* were introduced by Sederberg et al. as a generalization of NURBS surfaces [6,7]. The control meshes of T-splines permit T-junctions, which allows T-splines locally refinable without propagating entire columns or rows. This property makes T-splines an ideal modeling tool for complicated geometry. However, T-splines blending functions are not always linearly independent, which limits its applications in analysis. A solution to this problem is the so called *analysis-suitable T-splines* [8] which are a subset of T-splines defined over a restricted T-mesh whose T-junction extensions do not intersect. The blending functions of AST splines are always linearly independent and thus is suitable for finite element analysis. Other types of local refinement splines proposed in recent years include *PHT-splines* [9] and *LR-splines* [10], etc. They also have good properties which are ideal in CAD and iso-geometric analysis, for more details we the reader refer to [9,10].

An attractive alternative of tensor-product representation is to define piecewise polynomials on triangular partitions, since a triangulation has more flexibility to be adapted to arbitrary shapes. Traditional finite element spaces are defined on conforming triangulation, and local refinement must guarantee the conformability of the triangulation. Furthermore, to construct smooth (at least C^1 continuous) finite elements, generally high degree polynomials are required (for example, Argyris element consists of quintic polynomials with C^1 continuity globally) or each triangle is further subdivided into many sub-triangles (e.g., for Powell–Sabin elements, each macro-triangle is subdivided into 6 sub-triangles [11]). Recently, Speleers et al. constructed hierarchical Powell–Sabin splines for iso-geometric analysis applications [12,13]. Another related work is hierarchical triangular splines (HTS) introduced in [14]. A HTS spline surface is a piecewise quintic Bézier surface and is overall tangent plane continuous. The local refinement can be done by splitting each of the refined triangles into four sub-triangles regularly, which is referred to as a macro-patch. The Bézier ordinates on a macro-patch are constrained by G^1 continuity. HTS mainly focuses on modeling instead of analysis.

In this paper, we extend HB-splines paradigm to bivariate splines defined on regular triangulation. Starting from a spline space over a regular triangulation, we construct a nested spline space sequences according to the successive refinements of the regular triangulation by taking a similar approach as in [4]. This construction can be easily generalized to the spline spaces defined on other triangulation if local support basis functions of the space can be constructed. In this paper, we will focus our attention on spline spaces $S_3^1(\Delta_{mn}^{(1)}) - C^1$ continuous cubic splines over type-I triangulations, and $S_2^1(\Delta_{mn}^{(2)}) - C^1$ continuous quadratic splines over type-II triangulations which will be defined in details in Section 3.2.

The remainder of the current paper is organized as follows. In Section 3, we review some preliminary knowledge about bivariate splines defined on type-I or type-II triangular partition. In Section 4, the construction of hierarchical bivariate splines is described. Some properties of hierarchical bivariate spline basis functions are discussed. In Sections 5 and 6, applications of hierarchical bivariate splines in surface fitting and finite element analysis are demonstrated. Section 7 concludes the paper with a summary and future work.

3. Bivariate splines space defined on type-I and type-II triangular partitions

In this section, we recall some preliminary knowledge about bivariate splines defined on type-I and type-II partitions which have been thoroughly discussed in [2]. Without of generality, we assume that the initial domain $D = [0, m] \otimes [0, n]$ is a rectangle domain, here m, n are positive integers. L-type domain can be segmented into several rectangles. We should note that any linear transformation of the domain does not influence the results in this section, so D can be any parallelogram.

3.1. Splines space defined on type-I triangular partition

Type-I triangular partition, denoted by $\Delta_{mn}^{(1)}$, is constructed by connecting the diagonal line segment with positive slope of every rectangular cell of a uniform rectangular partition, see Fig. 1(a):

$$\Delta_{mn}^{(1)} : x = i, y = j, \; x - y = h,$$

where $i = 1, \ldots, m-1, \; j = 1, \ldots, n-1$, and $h = -n+1, \ldots, m-1$.

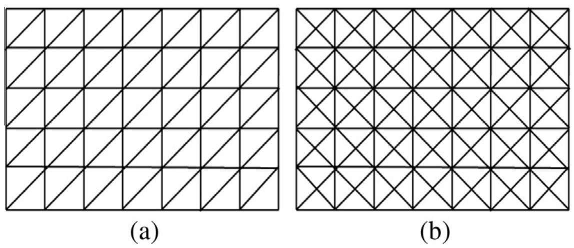

Fig. 1. (a) Type-I triangular partition. (b) Type-II triangular partition.

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