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Anisotropic surface meshing with conformal embedding

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ABSTRACT

This paper introduces a parameterization-based approach for anisotropic surface meshing. Given an input surface equipped with an arbitrary Riemannian metric, this method generates a metric-adapted mesh with user-specified number of vertices. In the proposed method, the edge length of the input surface is directly adjusted according to the given Riemannian metric at first. Then the adjusted surface is conformally embedded into a parametric 2D domain and a weighted Centroidal Voronoi Tessellation and its dual Delaunay triangulation are computed on the parametric domain. Finally the generated Delaunay triangulation can be mapped from the parametric domain to the original space, and the triangulation exhibits the desired anisotropic property. We compute the high-quality remeshing results for surfaces with different types of topologies and compare our method with several state-of-the-art approaches in anisotropic surface meshing by using the standard measurement criteria.

1. Introduction

Nowadays, anisotropic meshes are important for improving the accuracy of the numerical simulations as well as better approximating the shapes [1,2]. The anisotropic meshes are designed as elongated mesh elements with desired orientations and aspect ratios as user specified. These types of meshes are used in movies, animations, computer-aided design (CAD), computer-aided manufacturing (CAM), architecture design, and scientific visualization. They can provide more accurate approximations for the surface of the original object than the correspondent isotropic counterpart, in regions where the magnitudes of two principal curvatures are different. To accurately simulate the behavior of the physical phenomena, such as the flow of water, air across the earth, the deformation and wrinkles of clothes, the anisotropic meshes are preferred. In this paper, our goal is to generate the anisotropic triangle mesh with stretching ratios and directions conforming to the user-specified metric tensor field.

One typical way to generate the anisotropic triangular mesh is to compute Anisotropic Centroidal Voronoi Tessellation (ACVT) [3,4] and its dual mesh. This method needs to compute an Anisotropic Voronoi Diagram (AVD) in the ambient 3D space and its intersection of a surface, with sites constrained on the surface. This technique may lead to disconnected Voronoi cells without the topology control, if two regions are very close in 3D space but are far away along the surface. Especially with large anisotropic stretching ratios, the computed constrained ACVT on surface tends to be incorrect. It is natural to use the geodesic distance to compute the constrained ACVT, but the computation of the geodesic distance is difficult to be accurate and efficient.

To overcome the above limitations, we propose a new method for anisotropic meshing of surfaces endowed with Riemannian metrics. Theoretically, the best way for

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anisotropic mesh generation is to compute a perfect isometric embedding of the Riemannian surface into a high-dimensional Euclidean space [5], where the metric becomes uniformly identity, and then directly compute Centroidal Voronoi Tessellation (CVT) on it. However, it is very challenging to find such a high-dimensional isometric embedding – so far we have not seen any method available to compute it. So we take an alternative approach: instead of isometrically embedding the Riemannian surface into a high-dimensional space, we "conformally" embed the surface into a 2D parametric domain, followed by a weighted CVT computation on such 2D parametric domain. Such conformal embedding is achieved by: (1) adjusting only the edge length of the input surface according to the given Riemannian metric, without explicitly computing the vertex coordinates (i.e. its embedding); (2) conformally deforming the edge length so that the surface can be embedded into a 2D parametric domain. Here the conformal embedding tries to preserve the angle of the input surface triangles, thus it preserves the local aspect-ratios of the input metric. Then we rely on the computation of weighted CVT on this 2D parametric domain with a density function applied to compensate for the introduced area distortion of surface parameterization.

An advantage of this method is its efficiency since the CVT computation is performed on a 2D parametric domain, as compared to the traditional approach of computing ACVT and its intersection with the surface in 3D space [3,4]. Finally the dual triangulation of computed CVT is mapped back to the original domain and an anisotropic meshing result is gained. In Section 5 we show the comparison with existing approaches in anisotropic surface meshing by using the standard measurement criteria.

2. Backgrounds and related works

2.1. Anisotropic metric

Anisotropy defines the distortion of the distances and angles. Consider the domain $\Omega \subset \mathbb{R}^m$, and a given point $\mathbf{x} \in \Omega$ with endowed metric $\mathbf{M}(\mathbf{x})$. The anisotropic squared length of a vector $\mathbf{a}$ at $\mathbf{x}$ can be measured by the dot product between $\mathbf{a}$ and itself, using the metric $\mathbf{M}(\mathbf{x})$ as follows:

$$\|\mathbf{a}\|^2_{\mathbf{M}(\mathbf{x})} = \langle \mathbf{a}, \mathbf{a} \rangle_{\mathbf{M}(\mathbf{x})} = \mathbf{a}^T \mathbf{M}(\mathbf{x}) \mathbf{a}. \tag{1}$$

The metric matrix $\mathbf{M}(\mathbf{x})$ is a symmetric positive-definite (SPD) $m \times m$ matrix, which can be decomposed with Singular Value Decomposition (SVD) into:

$$\mathbf{M}(\mathbf{x}) = \mathbf{R}(\mathbf{x})^T \mathbf{S}(\mathbf{x})^2 \mathbf{R}(\mathbf{x}), \tag{2}$$

where the diagonal matrix $\mathbf{S}(\mathbf{x})^2$ contains its ordered eigenvalues, and the orthonormal matrix $\mathbf{R}(\mathbf{x})$ contains its eigenvectors. If we denote $\mathbf{Q}(\mathbf{x}) = \mathbf{S}(\mathbf{x})\mathbf{R}(\mathbf{x})$, by combining Eqs. (1) and (2), we can understand the anisotropic squared length of vector $\mathbf{a}$ as the "isotropic" squared length of its transformed vector $\mathbf{b} = \mathbf{Q}(\mathbf{x})\mathbf{a}$, as:

$$\|\mathbf{b}\|^2_{\mathbf{I}} = \mathbf{b}^T \mathbf{b} = \mathbf{a}^T \mathbf{M}(\mathbf{x}) \mathbf{a} = \|\mathbf{a}\|^2_{\mathbf{M}(\mathbf{x})}, \tag{3}$$

where $\mathbf{I}$ is the identity matrix. This means the vector $\mathbf{a}$ is rotated by $\mathbf{R}(\mathbf{x})$ and then scaled by $\mathbf{S}(\mathbf{x})$, before measuring its Euclidean length.

For the experiments given in this paper (Section 4), users start from designing a smooth scaling field $\mathbf{S}(\mathbf{x})$ and a rotation field $\mathbf{R}(\mathbf{x})$ that is smooth in regions other than some singularities, and then compose them to $\mathbf{Q}(\mathbf{x}) = \mathbf{S}(\mathbf{x})\mathbf{R}(\mathbf{x})$ and $\mathbf{M}(\mathbf{x}) = \mathbf{Q}(\mathbf{x})^T \mathbf{Q}(\mathbf{x})$, which is similar to the way of Du et al. [3]. The metrics are defined on the tangent spaces of the surface. Suppose s_1 and s_2 are the two diagonal items in $\mathbf{S}(\mathbf{x})$ corresponding to the two eigenvectors in the tangent space, and $s_1 \leqslant s_2$. Then we can define the *stretching ratio* as $\frac{s_2}{s_1}$ [6]. For the anisotropic meshing on 3D surfaces, we use the following metric tensor:

$$\mathbf{M} = [\mathbf{v}_{min}, \mathbf{v}_{max}, \mathbf{n}] diag({s_1}^2, s_2^2, 0) [\mathbf{v}_{min}, \mathbf{v}_{max}, \mathbf{n}]^T, \tag{4}$$

where $\mathbf{v}_{min}$ and $\mathbf{v}_{max}$ are the directions of the principal curvatures, $\mathbf{n}$ is the unit surface normal. s_1 and s_2 are two user-specified stretching factors along principal directions. To ensure smoothness of the input metric field, as suggested by Alliez et al. [7], we apply Laplacian smoothing to both the stretching ratios and directions.

2.2. Universal covering space and its mappings

Let Σ be a surface embedded in R^3. Let $\mathbf{g}$ be the Riemannian metric of Σ induced from its Euclidean metric in R^3. Suppose $u : \Sigma \to R$ is a scalar function defined on Σ. Then $\bar{\mathbf{g}} = e^{2u}\mathbf{g}$ is also a Riemannian metric on Σ. Angles measured by $\mathbf{g}$ are equal to those measured by $\bar{\mathbf{g}}$. Therefore, we say $\bar{\mathbf{g}}$ is conformal to the original metric.

According to the Uniformization Theorem [8], any surface admits a Riemannian metric of a constant Gaussian curvature, which is conformal to the original one. Such metric is called the uniformization metric. Specifically, surfaces with positive Euler characteristics (i.e., $\chi > 0$) exist spherical uniformization metric with +1 Gaussian curvature everywhere. Surfaces with zero Euler characteristics (i.e., $\chi = 0$) exist Euclidean uniformization metric with 0 Gaussian curvature everywhere. Surfaces with negative Euler characteristics (i.e., $\chi < 0$) exist hyperbolic uniformization metric with −1 Gaussian curvature everywhere.

A universal covering space of a surface, explained in an intuitive and informal way, is a simply connected periodic tessellation of the surface domain. Locally, the mapping between the surface and its universal covering space is one-to-one and continuous. Based on the uniformization metric, the universal covering space of a closed surface can be isometrically embedded into one of the three canonical domains: the spherical domain $\mathbb{S}^2$ for genus zero surfaces with positive Euler numbers, the planar domain $\mathbb{E}^2$ for genus one surfaces with zero Euler number, and the hyperbolic space $\mathbb{H}^2$ for high genus surfaces with negative Euler numbers (see Fig. 1).

In particular, our work appears to be mainly an extension of the techniques in [9,10] to the anisotropic setting. Additionally, the use of conformal parameterization for anisotropic remeshing has been mentioned in [11], but it takes topological disk as input that means any given