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Blow-up phenomena for a nonlocal semilinear parabolic equation with positive initial energy

Ali Khelghati *, Khadijeh Baghaei

Department of Mathematics, Payame Noor University (PNU), P.O. Box 19395-3697, Tehran, Iran

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ABSTRACT

This paper is concerned with the blow-up of solutions to the following semilinear parabolic equation:

$$u_t = \Delta u + |u|^{p-1}u - \frac{1}{|\Omega|}\int_\Omega |u|^{p-1}u\,dx, \quad x \in \Omega,\ t > 0,$$

under homogeneous Neumann boundary condition in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, with smooth boundary.

For all $p > 1$, we prove that the classical solutions to the above equation blow up in finite time when the initial energy is positive and initial data is suitably large. This result improves a recent result by Gao and Han (2011) which asserts the blow-up of classical solutions for $n \geq 3$ provided that $1 < p \leq \frac{n+2}{n-2}$.

1. Introduction

In this paper, we study the blow-up phenomena for the following semilinear parabolic problem:

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega |u|^{p-1}u\,dx, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu}(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded domain with smooth boundary, $\frac{\partial}{\partial \nu}$ denotes differentiation with respect to the outward normal ν on $\partial\Omega$ and $p > 1$. Moreover, we assume that $u_0(x)$ belongs to $C^{2,\sigma}(\bar{\Omega})$ with $0 < \sigma < 1$ and satisfies the integral constraint

$$\int_\Omega u_0(x)\,dx = 0 \quad \text{with } u_0(x) \not\equiv 0. \tag{1.2}$$

By considering the standard theory of parabolic equations, the local existence and uniqueness of solutions to problem (1.1)–(1.2) is clear. We refer the reader to [1] for details.

* Corresponding author.
E-mail addresses: khelghatibashkand@gmail.com (A. Khelghati), kh.baghaei@gmail.com (K. Baghaei).

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By integrating Eq. (1.1) and using integration by parts, it follows that

$$\begin{aligned}\frac{d}{dt}\int_\Omega u(x,t)dx &= \int_\Omega u_t(x,t)dx\\ &= \int_\Omega\Big(\Delta u + |u|^{p-1}u - \frac{1}{|\Omega|}\int_\Omega |u|^{p-1}u\,dx\Big)dx\\ &= 0.\end{aligned}$$

Thus we get

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx = 0. \tag{1.3}$$

This shows that the integral of u is conserved.

Problem (1.1) appears frequently in nuclear sciences when the growth of temperature is very fast and the total mass is conserved. It can also be used in chemistry and biological sciences where the total mass of a chemical or an organism is conserved. Other examples of nonlocal reaction–diffusion equations are given in [2,3].

The authors in [4] considered problem (1.1) with the constraints $u_0 \in C^3(\bar{\Omega})$ and $\int_\Omega u_0(x)dx = 1$. They showed that solutions of the problem (1.1) blow up in finite time if $p > 1$ and $J_{u_0} := -E(u_0)$ is suitably large, where $E(u_0)$ is defined as follows:

$$E(u_0) = \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx - \frac{1}{p+1}\int_\Omega |u_0|^{p+1}dx.$$

El Soufi et al. in [5] studied the equation

$$u_t = \Delta u + |u|^p - \frac{1}{|\Omega|}\int_\Omega |u|^p\,dx$$

with the same initial and boundary conditions are given in problem (1.1)–(1.2). For $1 < p \le 2$, they obtained a relationship between the finite time blow-up of solutions and the negativity of the initial energy. Later, in [6], the authors extended these results and proved that for $p > 1$ and $E(u_0) \le 0$, the solution does not exist in $L^\infty((0,T); L^2(\Omega))$ for all $T > 0$.

The authors in [7] investigated the blow-up phenomena for solutions of (1.1)–(1.2) with the positive initial energy. They constructed a control function and applied suitable embedding theorems to prove that the classical solutions to the problem (1.1) blow up in finite time provided that $E(u_0) < E_1$ and $\|\nabla u_0\|_2 > \beta$, where E_1 and β are positive constants which are introduced in their work. In addition, p must satisfy in the following conditions:

$$\begin{cases} 1 < p \le \infty, & \text{if } n = 1, 2,\\ 1 < p \le \dfrac{n+2}{n-2}, & \text{if } n \ge 3.\end{cases}$$

They also proposed a conjecture that the blow up in finite time may occur for all $p > 1$ and $n \ge 3$. In this paper, we give one answer to this conjecture. We prove that for all $p > 1$ and arbitrary positive initial energy, the classical solutions to the problem (1.1)–(1.2) blow up in finite time when the initial data is suitably large. In the next section we will prove the above result.

2. Blow up in finite time

In this section, we prove the blow-up of solutions to problem (1.1)–(1.2) with arbitrary positive initial energy and suitable initial data. The main method employed in this paper is based on the calculation of the energy functional and concavity argument developed by Levin [8]. In order to prove our main result, we will use the following lemma.

Lemma 2.1 ([9, Lemma 1.1]). *Suppose that a positive, twice-differentiable function $\theta(t)$ satisfies the inequality*

$$\theta''(t)\theta(t) - (1+\beta)\theta'^2(t) \ge 0, \quad t > 0,$$

where $\beta > 0$ is some constant. If $\theta(0) > 0$ and $\theta'(0) > 0$, then there exists $0 < t_1 \le \frac{\theta(0)}{\beta\theta'(0)}$ such that $\theta(t)$ tends to infinity as $t \to t_1$.

Here, we have our main result.

Theorem 2.2. *Let $u(x,t)$ be a classical solution to problem (1.1)–(1.2) in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \ge 1$. Then for all $p > 1$ and $E(u_0) > 0$, the solution $u(x,t)$ blows up in finite time provided that*

$$\|u_0\|_2^2 > \left(\frac{2(p+1)E(u_0)}{p-1}\right)^{\frac{2}{p+1}} |\Omega|^{\frac{p-1}{p+1}}. \tag{2.1}$$