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# Inverse spectral problem for the density of a vibrating elastic membrane

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ABSTRACT

This paper is concerned with the recovery of an unknown symmetric density function in the weighted Helmholtz equation with Dirichlet boundary conditions from the lowest few eigenvalues. By using the piecewise constant function to approximate the density function and using the Rayleigh–Ritz approach to discretize the differential equation, the continuous inverse eigenvalue problem is converted to a related matrix inverse eigenvalue problem and then a least squares problem for the discrete model is formulated. The solution of the least squares problem via an iterative method is discussed and then an approximation to the unknown density is recovered. Numerical experiments are given to confirm its competitiveness.



## 1. Introduction

In this paper, we are concerned with the two-dimensional inverse spectral problem for the clamped membrane. We look for a way to recover the unknown density function $\rho > 0$ from a single set of eigenvalues of

$$\begin{aligned} -\Delta u &= \lambda\rho u \quad \text{in } R, \\ u &= 0 \quad \text{on } \partial R, \end{aligned} \tag{1}$$

where $R$ is the known rectangle $(0, \pi/a) \times (0, \pi)$. As in [1], we assume that the $m$ lowest eigenvalues of (1) are given, and the unknown density $\rho$ is symmetric with respect to the midlines of $R$ and is also a small perturbation of $\rho = 1$ in $L^\infty$. The direct problem is finding a real number $\lambda$ such that the boundary value problem (1) has a non-trivial solution for a given $\rho > 0$. When $\rho(x) > 0$ is smooth sufficiently, it is shown in [2] that the solution of (1) belongs to $C^{5,\sigma}(\bar{R})$ based on the well-known regularity theorem given in [3] for the solution of a uniform elliptic Laplace boundary value problem on a rectangle. Here, $0 < \sigma < 1$ and $C^{5,\sigma}(\bar{R})$ is the space of functions in $C^5(\bar{R})$ whose derivatives of order 5 are Hölder continuous. Also, it is pointed out in [4] that for $\rho > 0$, there exists a countable set of eigenvalues and the corresponding orthonormal system of eigenfunctions $\{u_n(x, y)\}$ $(n = 1, 2, \ldots)$ forms a basis in $L^2$ with the weight $\rho$.

The motivation for this work is the fact that the Helmholtz equation arises naturally in many scientific and engineering areas, and the inverse problems appears very naturally in various applications including biomedical imaging [5], impedance imaging [6], optical imaging for non-destructive evaluation, and wave propagation and scattering [7]. For example, the vibrating elastic membrane which can be described by (1) is a classical problem in mathematical physics. Typically the

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geometry of the membrane can be determined by inspection of the physical object. For instance, when a clamped drum is considered, the geometry of the membrane is described by the shape of the drum. Thus, it is critical to determine the nature of any non-homogeneity. As it is pointed out in [8], the eigenvalues of particular membranes are often available based on frequency measurements. Therefore, an approach for the recovery of the density from the available eigenvalues is of practical importance.

Compared with the one-dimensional problem [9–12], inverse spectral results for the two-dimensional problem are more difficult to establish. The literature on two-dimensional inverse spectral problems is not as much as that on the one-dimensional case. The reader is referred to [13–16,8,1,17–19] for two-dimensional inverse spectral problems.

Of particular interest to this paper is the method proposed by C.M. McCarthy [1] for the recovery of the symmetric density on a rectangle from the $m$ lowest eigenvalues and the unknown density $\rho$ is assumed to be a small perturbation around 1 in $L^\infty$. The method in [1] is an extension of the one for the two-dimensional inverse Sturm–Liouville problem for the potential on a rectangle with Dirichlet boundary conditions [15] while the method in [15] is an extension of the one for the one-dimensional inverse Sturm–Liouville problem for the potential [20]. All the papers of [20,15,1] rely on the use of a finite dimensional matrix inverse eigenvalue problem to approximate the continuous inverse problem. This kind of conversion is usually given as motivation for work on the matrix inverse eigenvalue problem [21,22].

In [1], by expanding the function $\rho - 1$ with $m$ Fourier-type functions and discretizing the differential equation with Rayleigh–Ritz approach, the continuous inverse eigenvalue problem is converted to a matrix inverse eigenvalue problem with $m$ unknown parameters. When eigenvalues of (1) with $\rho = 1$ are all simple, the matrix inverse eigenvalue problem is solved by forming an associated set of nonlinear equations and solving these by a fixed-point iterative approach. Then an approximation to the unknown density $\rho$ is constructed. When some eigenvalues of (1) with $\rho = 1$ are multiple, the algorithm is modified. However, as pointed out in [1], the modification may fail since its success depends heavily on the symmetric property of the eigenfunctions of (1) with $\rho = 1$, and additional spectral data, such as symmetry properties of eigenfunctions, must then be specified in order to select the correct construction. Numerical results in [1] suggest that when $\rho - 1$ has compact support, expanding $\rho - 1$ with $m$ Fourier-type functions leads to some error near the boundary since the Fourier-type functions do not have compact support.

The inverse problem is ill-posed, and it is hard to recover the coefficients from part of eigenvalues, so we only consider the $\rho$ that is a function around constant 1 as in [1]. To reduce the error near the boundary, in this paper we attempt to approximate $\rho$ by a piecewise constant and numerical experiments show that it is better than the Fourier series sometimes. By using the piecewise constant function to approximate the density $\rho$ and using the Rayleigh–Ritz approach to discretize the differential equation, we get a matrix inverse eigenvalue problem, and approximate the original continuous inverse eigenvalue problem by it. Then the piecewise constant approximation is obtained by solving the matrix inverse eigenvalue problem. Thus far, most algorithms discussed in the literature for solving the continuous inverse eigenvalue problem are developed under a prior assumption that a solution somewhat is known to exist. However, this piece of information usually is not available in practice. Therefore, in our paper, we put forward a least squares problem and solve it by an iterative method, and then the piecewise constant approximation to the density $\rho$ is produced. Numerical results in Section 4 for examples used in [1] show our method produces a good approximation to the unknown density $\rho$ and for $\rho - 1$ with compact support, the error near the boundary may be reduced. Also, numerical results indicate that the sequence generated by our method may converge to the correct approximation to the unknown density $\rho$ in the case that multiple eigenvalues of (1) with $\rho = 1$ occur while the success of the method in [1] depends on the symmetric property of the eigenfunctions.

The rest of this paper is organized as follows. In Section 2 the least squares formulation for the related matrix inverse eigenvalue problem is presented. In Section 3 the details of recovery of the symmetric density function by using an iterative method are given. Finally, Section 4 is devoted to reporting our numerical results, which show that the numerical values are more accurate than that in [1] for some examples and the sequence generated by our method may converge to the correct approximation to the unknown density in the case that multiple eigenvalues of (1) with $\rho = 1$ occur while in [1] it may be difficult to select the correct construction without additional spectral data.

## 2. Least squares formulation for the inverse eigenvalue problem

Given the $m$ lowest eigenvalues $\{\lambda_i\}_{i=1}^m$ of (1), we seek an approximation to $\rho$, which is assumed to be symmetric with respect to the midlines of the rectangle $R$, and to be a small perturbation of $\rho = 1$ in $L^\infty$ as in [1]. The eigenvalues and eigenfunctions for the base problem with $\rho = 1$, which can be written down explicitly for the case of a rectangular domain, play an important role in the recovery. Denote with $\{\lambda_i^0, \phi_i^0\}_{i=0}^\infty$ the eigenpairs of (1) for the problem with $\rho = 1$ and $\{\phi_i^0\}_{i=1}^\infty$ are $L^2$-orthonormal eigenfunctions. In [1], $\{\lambda_i^0\}_{i=1}^\infty$ and $\{\phi_i^0\}_{i=1}^\infty$ shall be referred to as the base eigenvalues and the base eigenfunctions, respectively, and they are given by

$$\lambda_i^0 = a^2 n_i^2 + m_i^2 \tag{2}$$

and

$$\phi_i^0 = \frac{2\sqrt{a}}{\pi} \sin(a n_i x) \sin(m_i y), \tag{3}$$