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Projection methods for two velocity–two pressure models for flows of heterogeneous mixtures

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ABSTRACT

In this paper, a numerical analysis of two velocity–two pressure models for flows of solid particles and fluids is presented. First, a formal exploitation of the weak formulation of such models asserts that they are amenable to integration via projection methods. The challenging issues in the algorithm development for these models are then documented and suitable numerical methodologies for their remedy are devised. Subsequently, an algorithm for the integration of the models of interest is proposed. This is a two-phase projection method on collocated grids that utilizes a fractional-step time-marching scheme. It is further endowed with an interface detection-and-treatment methodology to properly account for the stiffness induced by the presence of moving and deforming material interfaces. The efficiency and robustness of the proposed numerical method are assessed in a series of numerical experiments that are delineated in the last part of this paper.

1. Introduction

In the course of the last decades, algorithm development and numerical simulation of two-phase flows of immiscible mixtures has been the subject of numerous research efforts. The notable increase of interest in this particular research field can be summarized in three aspects. To begin with, such flows are omnipresent in both technological applications and in natural phenomena, so that their study is of profound importance and applicability. Secondly, numerical simulations are well-adapted and have compelling advantages for the systematic study of the flows of interest. For example, the effects of specific processes can be effortlessly isolated by simply "switching off" the terms that describe them in the governing equations. Further, detailed parametric studies, which allow for an holistic understanding of the mechanisms that drive these flows, can be performed in relatively short times; moreover the extraction of flow features is straightforward. Thirdly, the predictive capacity of numerical simulations is dependent on the accuracy of the algorithms employed for the integration of the corresponding mathematical models. Consequently, the design and development of efficient and robust numerical methods for the integration of the related mathematical models is fostered.

To date, the majority of studies have focused on the integration of the two-phase Navier–Stokes equations, which govern the motion of immiscible mixtures of fluids, such as water and air [1–4]. Besides the equations for the balance of mass and linear momentum, appropriately weighted to account for the different viscosities and densities of the constituents, the two-phase Navier–Stokes equations are also endowed with a transport equation that models the propagation of the material interface. Typically, quantities such as the density and viscosity are discontinuous across such interfaces which, in turn, introduces stiffness problems that have to be properly remedied. Therefore, the numerical treatment of the two-phase

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Navier–Stokes requires the combination of a "Navier–Stokes solver" with a methodology for the accurate computation of the motion of the interface.

On the other hand, the modelling of immiscible and heterogeneous mixtures is based on the constitutive assumption that the phases coexist in space, so that the introduction of the concept of the volume fraction is required. Typical examples include, but are not limited to, flows of solid particles immersed in fluids and of fluids over and through porous media. For the modelling of these flows, one has to employ two velocity–two pressure models, so that the dynamics of each constituent are taken into consideration. As a result, each phase is assigned its own set of equations for the balance of mass and linear momentum, with the latter being augmented to accommodate terms that model interactions between the two phases. For more information about the derivation of two velocity–two pressure models, the axiomatic framework that they are based on, and the mechanics of coexisting continua more generally we refer the reader to [5].

In recent years there has been an increasing interest in the algorithm development of two-velocity, two-pressure models; see, for example, the recent discussions of [6,7] that focus on the numerical integration of such models for granular flows. However, despite these efforts, relevant literature remains notably restrained and our understanding of the related numerical analysis is rudimentary. Accordingly, available numerical results typically correspond to fully developed, steady-state flows, [8–12].

In this paper, we are concerned with the development of projection methods for two velocity–two pressure models on collocated grids. By using a particular two-phase flow model for flows of fluid-saturated granular materials as a model of reference, we first formally demonstrate that this model is amenable to the numerical treatment via appropriately generalized projection methods. Subsequently, we identify the challenging issues that emerge in the spatial and temporal discretization of the governing equations and craft solutions for their remedy. Based on this analysis, we propose a two-phase generalization of the classical Chorin–Temam method, suitably tailored to the structure of the equations at hand. Following the earlier work of [7], the proposed algorithm is coupled with a regularization method that, besides its ease of implementation, can efficiently handle strong material interfaces with varying topology. In the last part of this study, the performance and efficacy of the proposed algorithm are assessed via the numerical simulation of a Poiseuille and Couette flow of a simple fluid over and through granular beds and of a transient flow of a simple fluid over two-dimensional dunes.

In view of the dearth of relevant studies in the existing literature, the objective and novelty of the present work is twofold. On the one hand, to systematically develop an efficient and robust multi-phase projection method that is well-adapted to the integration of two velocity–two pressure models. On the other hand, to devise a detailed "numerical guide" that collectively identifies and addresses the challenging issues that arise in the numerical analysis of such models that are currently not-well understood. As such, the present study aims to perform a first step towards the development of a comprehensive numerical analysis of this type of models. Our focus is here placed on the flow model of [13]; nevertheless, both the analysis presented herein and the results extend to a large class of similar models such as, for example, the ones of [8,9,14] and others.

2. Notation and preliminaries

Before proceeding to the main body of this paper, we specify the notation and terminology that will be employed in the following. Throughout this paper, Ω denotes a bounded and Lipschitz domain of $\mathbb{R}^3$ whereas $\mathbf{n}$ and $\boldsymbol{\tau}$ designate the unit vectors normal and tangential to $\partial\Omega$, respectively. Further, $\mathbf{e}_i$, $i = 1, \ldots, 3$ denotes the ith versor of the Cartesian coordinate system, *i.e.* the set $\{\mathbf{e}\}_{i=1}^{3}$ is the standard orthonormal basis of $\mathbb{R}^3$. Also, as usual, $C^m(\Omega)$, $m \in \mathbb{Z}^+$ stands for the space of functions that have m continuous derivatives in Ω.

The space $L^2(\Omega)$ stands for the space of square Lebesgue integrable functions in Ω, equipped with its usual norm, $\|f\|_{H^1(\Omega)} = \left(\int_\Omega |f|^2 \mathrm{d}\mathbf{x}\right)^{1/2}$. Further, the Sobolev space $H^1(\Omega)$ is defined as follows,

$$H^1(\Omega) = \left\{ f : \int_\Omega |f|^2 + |\nabla f|^2 \mathrm{d}\mathbf{x} < \infty \right\}, \tag{1}$$

and is endowed with the norm $\|f\|_{H^1(\Omega)} = \left(\int_\Omega |f|^2 \mathrm{d}\mathbf{x}\right)^{1/2} + \left(\int_\Omega |\nabla f|^2 \mathrm{d}\mathbf{x}\right)^{1/2}$. Let $C_c^\infty(\Omega)$ denote the space of smooth functions compactly supported in Ω. Then, $H_0^1(\Omega) \equiv \overline{C_c^\infty(\Omega)}^{\|\,\|_{H^1(\Omega)}}$, *i.e.* $H_0^1(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ with respect to the topology induced by the H^1-norm.

Consider the space of smooth and compactly supported solenoidal vector fields in Ω, $J(\Omega)$. We define $H(\Omega)$ as the *completion* of $J(\Omega)$ with respect to the norm induced by the inner product,

$$\langle u, v \rangle = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\mathbf{x}.$$

Finally, fix $0 < T \leq \infty$. Presume that $A(\Omega)$ and $B([0, T])$ are Banach spaces with norms $\|\ \|_{A(\Omega)}$ and $\|\ \|_{B([0,T])}$, respectively. The Bochner space $B([0, T] : A(\Omega))$ comprises functions $f \in A(\Omega)$ such that $\|f\|_{A(\Omega)} \in B([0, T])$ and is endowed with the norm $\|f\|_{B([0,T]:A(\Omega))} = \|\, \|f\|_{A(\Omega)} \|_{B([0,T])}$.

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