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# General spline filters for discontinuous Galerkin solutions

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ABSTRACT

The discontinuous Galerkin (dG) method outputs a sequence of polynomial pieces. Post-processing the sequence by Smoothness-Increasing Accuracy-Conserving (SIAC) convolution not only increases the smoothness of the sequence but can also improve its accuracy and yield superconvergence. SIAC convolution is considered optimal if the SIAC kernels, in the form of a linear combination of B-splines of degree $d$, reproduce polynomials of degree $2d$. This paper derives simple formulas for computing the optimal SIAC spline coefficients for the general case including non-uniform knots.



## 1. Introduction

The Discontinuous Galerkin (dG) method is widely used to approximately solve the weak formulation of partial differential equations. The lack of continuity between the dG elements models weak constraints between elements and is computationally convenient: discontinuity allows for a flexible discretization of the partial differential equations by locally adjusting the polynomial degree and element spacing; and the discontinuity increases opportunities for parallelism when stepping forward in a simulation. However, except near jump discontinuities, the inter-element discontinuities often do not agree with the expected smoothness of the outcome and hinders downstream applications such as stream line tracing [1].

Filtering, in particular Smoothness-Increasing Accuracy-Conserving (SIAC) filtering, has been proposed to smooth dG output while maintaining the order of the accuracy of the original dG solution. Remarkably, such post-filtering by convolution, can improve the accuracy of the resulting approximation as a solution to the partial differential equations, not only for dG, but also for other Galerkin-projection methods. Already Bramble and Schatz [2] showed that, for a wide class of elliptic boundary value problems and uniform subdivision of the domain, averaging of the output can yield superconvergence, i.e. a more accurate approximation to the solution than the degree of the elements suggests. Superconvergence is possible since certain integral norms, called moment norms [3] or negative-order norms, converge fast and this can be used to bound the error of the convolved output. In the context of linear hyperbolic partial differential equations this fact was convincingly demonstrated in [4].

Starting with [5], a series of papers has generalized SIAC filtering of dG output from the prototypical case of linear equations with periodic boundary conditions on a uniform mesh to non-uniform meshes and spatial dimensions two and three, including structured and unstructured bivariate and tetrahedral meshes [6–8]. A recent advance, presented in the minisymposium on post-processing dG solutions [9] organized by Mirzargar and Ryan, are simple formulas that allow solving for the coefficients of optimal SIAC spline filters with uniform knots. By contrast, [10] had to resort to Gaussian

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quadrature to determine the entries of the corresponding constraint matrix. Since the SIAC approach itself has recently been extended to non-uniform meshes, formulas corresponding to B-splines with non-uniform knot spacing deserve attention. This paper is first to derive the formulas for this case, and prove uniqueness.

**Overview.** Section 2 reviews, to the extent needed for the results, B-splines, SIAC filtering and convolution. Section 3 derives the entries of the constraint matrix whose solution yields the optimal coefficients for filters that post-process dG solutions with splines over non-uniform knot sequences. Section 4 discusses special choices of knot sequences.

## 2. Convolution and B-splines

The goal of SIAC filtering is to smooth out the sequence of polynomial pieces $p_j, j = 0..n$ on consecutive intervals $[t_j..t_{j+1}) \subsetneq \mathbb{R}$ that are output by dG computations. To this end, we will convolve the sequence with a linear combination of B-splines. The convolution $f * g$ of a function $f$ with a function $g$ is defined as

$$(f * g)(x) := \int_{\mathbb{R}} f(t)g(x-t)dt = (g * f)(x), \tag{1}$$

for every $x$ where the integral exists. When $g \geq 0$ and $\int_{\mathbb{R}} g = 1$ then the convolution has special, desirable properties: if $f$ is non-negative, (directionally) monotone or if $f$ is convex then so is $f * g$. Moreover, the graph of $f * g$ is in the convex hull of the graph of $f$. Convolution is commutative, associative and distributive.

A succinct but comprehensive treatment of B-splines can be found in Carl de Boor's summary [11] (see also [12]). There are a number of alternative ways to derive and define B-splines, for example as the smoothest class of piecewise polynomials over a given support. The subclass of uniform B-splines can alternatively be defined via convolution (efficiently carried out in Fourier space), a definition that is handy when deriving optimal coefficients of SIAC spline filters with *uniform* knot spacing. For general, non-uniform filters, the classical definition of splines via divided differences [13] is more convenient. We use the notation $i:j$ to abbreviate the sequence $i, i+1, \ldots, j-1, j$ and $t_{i:j}$ correspondingly to denote the sequence of real numbers $t_i, t_{i+1}, \ldots, t_j$. For a sufficiently smooth univariate real-valued function $h$ with $k$th derivative $h^{(k)}$, divided differences are defined by[1]

$$\Delta(t_i)h := h(t_i), \quad \text{and for } j > i$$

$$\Delta(t_{i:j})h := \begin{cases} (\Delta(t_{i+1:j})h - \Delta(t_{i:j-1})h)/(t_j - t_i), & \text{if } t_i \neq t_j, \\ \dfrac{h^{(j-i)}}{(j-i)!}(t_i), & \text{if } t_i = t_j. \end{cases} \tag{2}$$

If $t_{i:j}$ is a non-decreasing sequence, we call its elements $t_\ell$ *knots* and the classical definition of the *B-spline of degree d with knot sequence* $t_{i:j}, j := i + d + 1$ is

$$B(x|t_{i:j}) := (t_j - t_i)\ \Delta(t_{i:j})(\max\{(\cdot - x), 0\})^d. \tag{3}$$

Here $\Delta(t_{i:j})$ acts on the function $h : t \to (\max\{(t-x), 0\})^d$ for a given $x \in \mathbb{R}$. Consequently, a B-spline is a non-negative piecewise polynomial function with support on the interval $[t_i..t_j)$. If $\mu$ is the multiplicity of the number $t_\ell$ in the sequence $t_{i:j}$, then $B(x|t_{i:j})$ is at least $d - \mu$ times continuously differentiable at $t_\ell$.

The Peano formula of the remainder term, when approximating a $C^k$-function by its $k$th order Taylor expansion, can be expressed in terms of a B-spline $B(t|t_{0:k})$ as (see e.g. [14])

$$\frac{1}{k!}\int_{\mathbb{R}} B(t|t_{0:k})g^{(k)}(t)dt = \Delta(t_{0:k})g. \tag{4}$$

An important step when deriving the constraint matrix for optimal SIAC spline coefficients is to re-interpret this formula as a convolution formula for B-splines with monomials.

**Lemma 2.1.** *For integers $k > 0$ and $\delta \geq 0$, and the alternating monomial $(-\cdot)^\delta : t \to (-t)^\delta$,*

$$\left(B(\cdot|t_{0:k}) * (-\cdot)^\delta\right)(x) = \binom{k+\delta}{k}^{-1} \Delta\, t_{0:k}(t-x)^{k+\delta}. \tag{5}$$

**Proof.** For fixed $x$, we choose the function $g$ in (4) to be $g := \binom{k+\delta}{\delta}^{-1}(\cdot - x)^{k+\delta}$. Then $g^{(k)}(t) = (k!)(t-x)^\delta$ and

$$\left(B(\cdot|t_{0:k}) * (-\cdot)^\delta\right)(x) \overset{(1)}{=} \int_{\mathbb{R}} B(t|t_{0:k})(-(x-t))^\delta dt = \frac{1}{k!}\int_{\mathbb{R}} B(t|t_{0:k})g^{(k)}(t)dt$$

$$\overset{(4)}{=} \Delta(t_{0:k})g = \binom{k+\delta}{k}^{-1} \Delta\, t_{0:k}(t-x)^{k+\delta}. \quad \square$$

[1] The survey [14] advertises the symbol $\Delta$ for divided differences over alternatives such as $[t_{i:j}]h$ or $h[t_{i:j}]$.