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The scaling conjugate gradient iterative method for two types of linear matrix equations[☆]

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ABSTRACT

In this paper, we propose a new iteration method which is based on the conjugate gradient method for solving the linear matrix equations of the form $A_iXB_i = F_i,\ (i = 1, 2, \ldots, N)$ and the generalized Sylvester matrix equation $AXB + CXD = E$. This method is compared with some existing methods in detail, such as gradient based iterative (GI) method and least squares iterative (LSI) method (Ding et al. 2010) proposed by Ding et al., also cyclic and simultaneous methods (Tang et al. 2014) given by Tang et al. Some numerical experiments demonstrate that the introduced iterative method is more efficient than the four existing methods.

1. Introduction

The linear matrix equations are very important research problem in the fields of computational mathematics and control and system theory and so on [1–3]. A lot of efforts are related to varieties of systems of matrix equations [4–30].

The matrix equations

$$A_iXB_i = F_i \quad (i = 1, 2, \ldots, N) \tag{1.1}$$

receive a lot of attention owing to the wide applications, where $A_i \in R^{p_i\times m},\ B_i \in R^{n\times q_i},\ F_i \in R^{p_i\times q_i}$ are known matrices and $X \in^{m\times n}$ is the matrix to be determined. Ding et al. have derived some fruitful research productions [31–34], especially, [34] studied the hierarchical principle based iterative methods in which the general coupled Sylvester matrix equations were regarded as the unknown matrices or parameters. The basic idea was derived from the well-known Jacobi and Gauss–Seidel iterations for $Ax = b$, by extending these methods they gained iterative solutions of matrix equation $AXB = F$. Xie et al. also utilized the hierarchical principle to solve some linear matrix equations [35]. Zhou et al. proposed gradient based iterative methods [36] to find the unique solution of the general coupled Sylvester matrix equations by the weighted least squares and the gradient search principle. Tang et al. considered the so-called cyclic and simultaneous iterative methods for the matrix equations $A_iXB_i = F_i$ [37].

The generalized Sylvester matrix equation

$$AXB + CXD = E \tag{1.2}$$

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is a hot research issues, where $A, C \in R^{m\times m}, B, D \in R^{n\times n}, E \in R^{m\times n}$ are constant matrices and $X \in^{m\times n}$ is the unknown matrix. Navarra et al. proposed a representation of the general solution of the matrix equations $A_1XB_1 = C_1, A_2XB_2 = C_2$ [38]. Liao et al. in [39] obtained the least squares solution with the minimum-norm for the matrix equations $AXB = C, GXH = D$. Mitra found the conditions for the existence of a solution and a representation of the common solution for the matrix equations $AX = C, XB = D$ [40]. In [41–43], iterative algorithms were constructed to solve the equation $AXB = C$. By Moore–Penrose generalized inverse, some necessary and sufficient conditions for the existence of the solution and the expressions of the matrix equation $AX + X^TC = B$ are obtained in [44]. In addition, Deng et al. studied deeply the consistent conditions and the general expressions about the Hermitian solutions of the linear matrix equation [27].

The conjugate gradient method is known as the powerful method for solving the system of linear equations

$$Ax = b. \tag{1.3}$$

Many researchers have extended the method for solving matrix equations. Dehghan and Hajarian considered the generalized coupled Sylvester matrix equations $AXB + CYD = M, EXF + GYH = N$ and presented a modified conjugate gradient method to solve the generalized coupled Sylvester matrix equations on generalized bisymmetric matrix pair (X, Y) [45]. Liang and Liu proposed a modified conjugate gradient method to solve the equations $A_1XB_1 + C_1X^TD_1 = F_1, A_2XB_2 + C_2X^TD_2 = F_2$ [46]. In [47], the periodic Sylvester matrix equations $\widehat{A}_j\widehat{X}_j\widehat{B}_j + \widehat{C}_j\widehat{X}_{j+1}\widehat{D}_j = \widehat{E}_j$ $(j = 1, 2, \ldots)$ with the period solutions $\widehat{X}_{j+\lambda} = \widehat{X}_j$ were proposed, where λ denotes the period. The Kalman–Yakubovich conjugate matrix equation $XF - A\overline{X} = C$ was proposed in [48]. Wu et al. in [49] presented a finite iterative method for the generalized Sylvester-conjugate matrix equation $AX + BY = E\overline{X}Y + S$.

In this paper, we present the scaling conjugate gradient (SCG) method to investigate the matrix equations (1.1) and the generalized Sylvester matrix equation (1.2).

For the convenience of our statements, we use the following notation throughout the paper: Let $\mathrm{R}^{m\times n}$ denotes the set of $m \times n$ real matrix. For $A \in R^{m\times n}$, we write A^T, $\|A\|$, $\|A\|_F$, $\rho(A)$ to denote the transpose, the spectral norm, the Frobenius norm, the largest eigenvalue, respectively. Let $x \in R^n$, $\|x\|_2$ denotes the usual Euclidean norm. For any matrix $A = (a_{ij}), B = (b_{ij})$, $A \otimes B$ denotes the Kronecker product defined as $A \otimes B = (a_{ij}B)$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. For the block-column matrix $X = (x_1, x_2, \ldots, x_n) \in R^{m\times n}$, vec$(X)$ denotes the vec operator defined as $\mathrm{vec}(X) = (x_1^T, x_2^T, \ldots, x_n^T)^T \in R^{mn}$. The inner product in space $R^{m\times n}$ is defined as

$$\langle A, B\rangle = \mathrm{tr}(A^TB), \tag{1.4}$$

especially, $\langle A, A\rangle = \mathrm{tr}(A^TA) = \|A\|_F^2$. The star $(*)$ product was introduced to denote the block-matrix inner product as [50].

Let

$$X := \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{pmatrix} \in R^{(mp)\times n}, \qquad Y := \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_p \end{pmatrix} \in R^{(np)\times m},$$

then the block-matrix star product is defined as

$$X * Y = \begin{pmatrix} X_1Y_1 \\ X_2Y_2 \\ \vdots \\ X_pY_p \end{pmatrix} \tag{1.5}$$

with $X_i, Y_i^T \in R^{m\times n}, i = 1, 2, \ldots, p$.

The present paper is organized as follows. We first recall GI method and LSI method and point out a mistake proof of Theorem 1 in [50] which is closely related to our paper. Then we give its correction proof in Section 2. In Section 3, we generalize the conjugate gradient (CG) method to the scaling conjugate gradient (SCG) method for solving the matrix equations (1.1). And we provide the convergence analysis in detail. In Section 4, we apply our new method to solve the generalized Sylvester matrix equation (1.2). Some Numerical tests are presented in Section 5 which show that the proposed approach is more efficient than the existing methods. In Section 6, we use a brief conclusion to end the paper.

2. The GI and LSI methods

In this section, we shall briefly review the basic ideas and the principles of the GI method and LSI method in [50]. Then we rectify the mistake in a major convergence theorem.

Let $M \in R^{n\times n}$ be a determined matrix and $\mu > 0$ be the convergence factor or step size. The iterative method for $Ax = b$ as follows [31]

$$x(k) = x(k-1) - \mu M(Ax(k-1) - b). \tag{2.1}$$

Clearly, it is the extension of the Jacobi and Gauss–Seidel iterations. If $A = L + D + U$, where L, D, U denote the strictly low triangular matrix, diagonal matrix, strictly upper triangular matrix, respectively. Taking $M = D^{-1}$ and $\mu = 1$, (2.1) reduces to the Jacobi iteration method. Similarly, let $M = (L + D)^{-1}$ and $\mu = 1$, one can get the Gauss–Seidel method.