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A tent pitching scheme motivated by Friedrichs theory ☆

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ABSTRACT

Certain Friedrichs systems can be posed on Hilbert spaces normed with a graph norm. Functions in such spaces arising from advective problems are found to have traces with a weak continuity property at points where the inflow and outflow boundaries meet. Motivated by this continuity property, an explicit space–time finite element scheme of the tent pitching type, with spaces that conform to the continuity property, is designed. Numerical results for a model one-dimensional wave propagation problem are presented.

1. Introduction

A commonly used approach for constructing numerical methods to solve time-dependent problems is based on the method of lines, where a discretization of all space derivatives is followed by a discretization of time derivatives. The resulting methods are called implicit or explicit depending on whether one can advance in time with or without solving a spatially global problem. The study in this paper targets a different class of methods referred to as *locally implicit* space–time finite element methods, which advance in time using calculations that are local within space–time regions of simulation. Examples of such methods are provided by "tent pitching" schemes, which mesh the space–time region using tent-shaped subdomains and advance in time by varying amounts at different points in space.

Ideas to advance a numerical solution in time by local operations in space–time regions were explored even as early as [1]. Recurrence relations on multiple slabs of rectangular space–time elements were considered in [2], whose ideas were generalized to non-rectangular space–time elements for beams and plates in [3]. These works are not so related to the current work as some of the more modern references. Closest in ancestry to the method we shall consider is found in [4] where it was called explicit space–time elements. The space–time discontinuous Galerkin (SDG) method was announced almost at the same time in [5] and continues to see active development [6–8]. Against this backdrop, we highlight two papers that brought tent pitching ideas into the numerical analysis community [9,10]. The questions we choose to ask in this work have been heavily influenced by these two works. We should note that the name "tent pitching" has been traditionally used for meshing schemes that advance a space–time front [11,12], but in this paper tent pitching refers to the discretization scheme together with all the required meshing.

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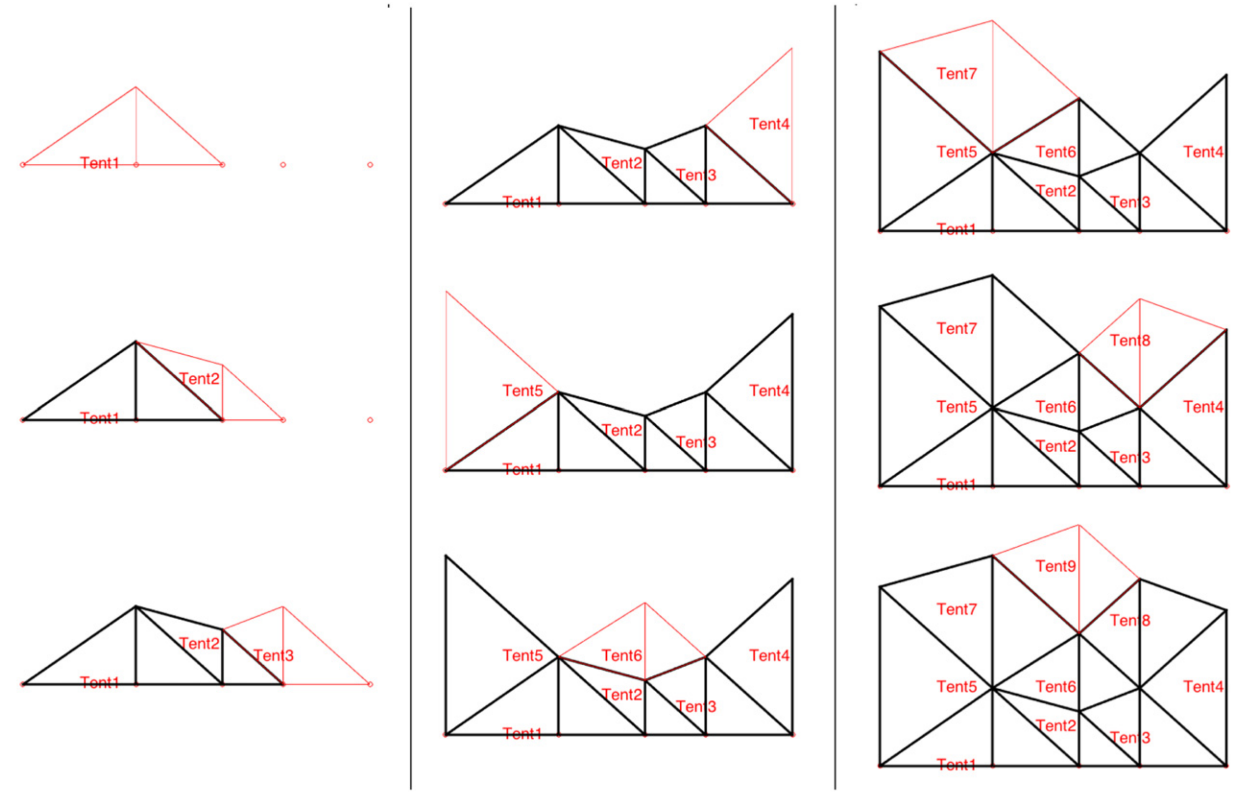

Fig. 1. Tent pitching (read column by column).

To give an overview of what is involved in a tent pitching scheme, consider the case of a hyperbolic problem posed in one space dimension with time as the second dimension. Given a spatial mesh, we pitch a tent by erecting a tent pole (vertically in time) at a vertex, as in Fig. 1. (Precise definitions of "tents", etc. appear later—see Definition 4.8.) In the plots of Fig. 1, the horizontal and vertical dimensions are space and time, respectively. The height of the tent pole must be chosen small enough in relation to the hyperbolic propagation speed, so that the domain of dependence of all points in the tent remains within the tent's footprint. We then use the given initial data to solve, by some numerical scheme, the hyperbolic problem restricted to the tent. Proceeding to the next vertex where the second tent is pitched in Fig. 1, we find that the initial data combined with the solution in the previous tent, provides inflow data to solve the hyperbolic problem there. Solution on the newer tents proceeds similarly. This shows the sense in which tent pitching schemes are locally implicit: they only involve solving local problems tent by tent.

Having explained tent pitching schemes in general, we should now emphasize that the main result of this paper is not a new tent pitching scheme (although one is included to show relevance). Rather, this paper is mainly concerned with answering a few theoretical questions *motivated* by tent pitching schemes. Indeed, our main result is a characterization of traces of a Friedrichs space on a tent-shaped domain and builds on the recent advances in Friedrichs theory [13–16]. To explain the Friedrichs connection, we should first note that all the previous tent pitching schemes use non-conforming space–time discontinuous Galerkin discretizations. Design of tent pitching methods within a conforming setting, while holding the promise of locally adaptive time marching with fewer unknowns, poses interesting questions: what is the weak formulation that the tent pitching scheme should conform to? What are the spaces? What are the finite element subspaces one should use? These questions form the motivation for this study and while attempting to answer them, Friedrichs spaces and their traces appear naturally, as we shall see. While we are far from answering the above questions for a general Friedrichs system, our modest aim in this paper is to provide some answers for a few simple problems in one space dimension.

Accordingly, there are two parts to this paper. The first and the main part of the paper consists of Sections 2–4. While results of Sections 2 and 3 are applicable to any abstract Friedrichs system, Section 4 focuses mainly on an advection example and its implications for hyperbolic systems. This leads to observations on the traces of certain Friedrichs spaces. The theory clarifies a weak continuity property of the traces at the points where inflow and outflow part of boundaries (defined precisely later) meet. It is relevant in the tent pitching context because in the tent-shaped domains used in tent pitching schemes, inflow and outflow boundaries always meet. The second part of the paper, consisting of Sections 5 and 6, designs an explicit space–time finite element scheme of the tent pitching type using the spaces and weak formulations motivated by the first