Computers and Mathematics with Applications 70 (2015) 1220–1233

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Computation of fixed boundary tokamak equilibria using a method based on approximate particular solutions

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ARTICLE INFO

Article history:
Received 5 December 2014
Received in revised form 9 July 2015
Accepted 14 July 2015
Available online 3 August 2015

Keywords:
Grad-Shafranov equation
Tokamak plasma equilibrium
Meshless method
Method of approximate particular solutions

ABSTRACT

In this work a meshless method based on the approximate particular solutions is applied to the computation of fixed boundary tokamak equilibria using Grad–Shafranov (GS) equation. The GS equation is solved for different choices of the right hand side of the equation: (i) when it is not a function of magnetic flux (i.e., Solov'ev solutions), (ii) when it is a linear function of magnetic flux, and (iii) when it is a nonlinear function of magnetic flux. For all these cases the first order derivative term in the GS equation is transferred to the right hand side such that the left hand side consists only the Laplace operator. This enables us to use the Radial Basis Functions (RBFs) in the calculation of approximate particular solutions. A linear combination of these particular solutions is taken as the solution of the GS equation and the resulting system of algebraic equations is solved iteratively because of the presence of the magnetic flux on the right hand side in all three choices. Furthermore, we use least squares approach in solving the overdetermined system of algebraic equations which alleviates the problem of ill-conditioning to a certain extent. The numerical results obtained using this method are in good agreement with the analytical solutions (where available). We find that the method is convergent, accurate and easily applicable to the irregular geometries due to its meshless character.

1. Introduction

The Magnetohydrodynamic (MHD) equilibria in the tokamak are described by the Grad–Shafranov (GS) equation [1]. This is in general a nonlinear elliptic partial differential equation derived from the ideal MHD equations. There are numerous extensive works in the literature for solving GS equation for the fixed and free boundary problems in the tokamak using the finite element, finite difference, spectral, boundary element and other mesh based methods [2–9]. However the work based on meshless methods applied to the GS equation is very limited. Some of these works are given in [10–12]. The meshless methods have the advantages that (i) they are comparatively easy to program, (ii) they do not require a mesh, and (iii) they can be applied to any geometry without much difficulty.

The aim of the present work is to formulate and apply a meshless method based on the approximate particular solutions for the computation of the fixed boundary tokamak equilibria. The governing equation consists of an axisymmetric operator on the left hand side of an elliptic partial differential equation. Although the GS operator is linear, to find its particular solution is very hard except for some cases when the right hand side is a combination of monomials or some other simple function [13]. Furthermore, if the approximate particular solutions are obtained by approximating the right hand side by radial basis functions (RBFs), then these particular solutions no longer remain purely radial functions, i.e., they contain both

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http://dx.doi.org/10.1016/j.camwa.2015.07.013

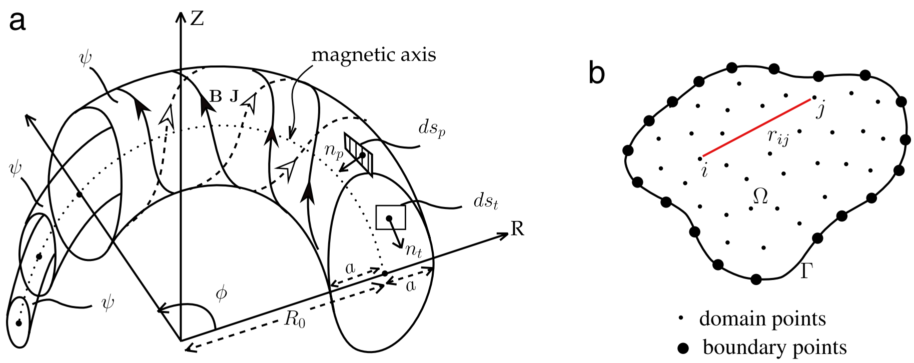

Fig. 1. (a) A torus showing flux surfaces (ψ = constant) on which **B** and **J** lie, (b) a schematic representation of domain and boundary on a poloidal cross-section.

coordinates (x and y) instead of being a function of radial distance only. In this work we rearrange the GS equation such that its left hand side consists of only the Laplace operator and the remaining term is transferred to the right hand side. This allows the particular solutions to be purely RBFs. The requirement of particular solutions to be RBFs is because the RBF interpolants possess certain important properties namely radial symmetry and invariance under translations, rotations, and reflections [14].

The linear combination of such particular solutions (associated with each node) is used to express the solution of the GS equation. It is noted that the fundamental solution which is needed for the calculation of homogeneous solution, as in the method of fundamental solutions (MFS) [15,12,16], is no longer needed in the present method. The present method was originally developed in [17] and named as MAPS (Method of Approximate Particular Solutions). We have applied it to the present problem in a least squares sense. The least squares method helps to alleviate the ill-conditioning (which is inherent in meshless methods) to a certain extent. The multiquadratic (MQ)-RBF has been used in the following calculations due to its better performance among the other globally supported RBFs [18,19].

In Section 2, the GS equation is presented in a nondimensional form. In Section 3, the least squares method based on the approximate particular solutions is presented to solve the GS equation numerically. In Section 4, the results are shown corresponding to the three choices of right hand side of the GS equation: (a) when it is not a function of magnetic flux (Ψ) (Solov'ev case), (b) when it is a linear function of Ψ, and (c) when it is a nonlinear function of Ψ. The conclusions are given in Section 5.

2. The Grad–Shafranov equation

The Grad–Shafranov equation is derived from the steady state ideal MHD equations in cylindrical coordinates (R, Z, ϕ) with the assumption of toroidal symmetry $(\partial/\partial\phi = 0)$ and static plasma ($\boldsymbol{v} = 0$) and can be written as follows [2,9]:

$$-\left\{R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\Psi}{\partial R}\right)+\frac{\partial^2\Psi}{\partial Z^2}\right\} = \mu_0 R^2\frac{dp(\Psi)}{d\Psi} + I(\psi)\frac{dI(\Psi)}{d\Psi} \equiv \mu_0 R J_\phi \tag{1}$$

where J_ϕ = toroidal component of the current density and μ_0 = magnetic permeability of free space. The magnetic flux Ψ in Eq. (1) is the poloidal flux Ψ_p normalized by 2π, i.e.,

$$\Psi = \frac{1}{2\pi}\int_{s_p}\mathbf{B}\cdot\mathbf{n}ds_p$$

where **B** is the magnetic field, **n** is the unit normal to a surface element ds_p as shown in Fig. 1(a). In Eq. (1) $p(\Psi)$ is the pressure as a function of Ψ, $I(\Psi)$ is the poloidal current function defined by $I(\Psi) = \int_{s_p}\mathbf{J}\cdot\mathbf{n}ds_p$ where **J** is the current density. The quantity $\Psi \equiv \Psi(R, Z)$ gives the equilibrium profile of the plasma in the geometry under consideration.

Nondimensionalisation

The GS equation (Eq. (1)) can be nondimensionalized using: $x = R/R_0$, $y = Z/R_0$, $\psi = \Psi/\Psi_0$, $B_x = B_R/\left(\Psi_0/R_0^2\right)$, $B_y = B_Z/\left(\Psi_0/R_0^2\right)$ and $j_\phi = \left(\mu_0 R_0^3/\Psi_0\right)J_\phi$, where R_0 is the major radius and Ψ_0 is an arbitrary value of the magnetic flux. In other words, using $R \to xR_0$, $Z \to yR_0$, $\Psi \to \Psi_0\psi$, $B_R \to \left(\Psi_0/R_0^2\right)B_x$, $B_Z \to \left(\Psi_0/R_0^2\right)B_y$ and $J_\phi \to \left(\Psi_0/\mu_0 R_0^3\right)j_\phi$, the GS equation can be written in the following nondimensional form:

$$-\triangle^*\psi \equiv -\left\{x\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial\psi}{\partial x}\right)+\frac{\partial^2\psi}{\partial y^2}\right\} = xj_\phi \equiv \mathcal{F}(\psi, x, y), \tag{2}$$