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Singular boundary method for heat conduction problems with certain spatially varying conductivity

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ABSTRACT

The singular boundary method (SBM) is a recent boundary-type meshless collocation method, in which the solution of a given problem is expanded as a linear combination of the fundamental solutions in terms of the source points. The method circumvents the fictitious boundary long perplexing the method of fundamental solution (MFS) by the introduction of the concept of origin intensity factors (OIFs). This paper documents the first attempt to extend the method to heat conduction problems in nonhomogeneous materials. We derive the fundamental solutions of heat conduction problems with the thermal conductivity of the quadratic, exponential and trigonometric material variations in three directions. Furthermore, we firstly theoretically derive the value of the OIF for the natural logarithm function, which is later extended to the OIFs for a group of fundamental solutions. The feasibility, accuracy and stability of the presented SBM formulation are confirmed for both two- (2D) and three-dimensional (3D) examples.

1. Introduction

In recent years, materials with mechanical properties that are continuous functions of spatial variables, for instance, functionally graded materials [1], draw the growing attention. Heat conduction analysis in these nonhomogeneous materials widely exists in various engineering applications. This is due to the fact that these materials permit tailoring of their microstructure to enhance their performance with respect to actual application demands.

However, the heterogeneity of the material poses a serious challenge to analytically solve this problem [2,3]. Numerical methods provide an efficient way to solve these types of problems. Traditionally, these problems are frequently solved by the finite element method, finite difference method, finite volume method, etc. These methods are well known as domain discretization techniques. Another attractive alternative that has been extensively used to solve a variety of engineering problems is the boundary element method (BEM). The BEM has long been considered as boundary-only discretizations owing to its semi-analytical nature with the fundamental solutions of the governing differential operator. Nevertheless, sophisticated mathematics is associated with the singular and nearly-singular integration.

In the past several decades, a group of fundamental solutions-based meshless collocation methods and their derivatives have been developed to eliminate the mesh connecting the data points of the simulation domain, for instance, the method of fundamental solutions (MFS) [4–7], singular boundary method (SBM) [8], boundary knot method (BKM) [9], boundary

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particle method (BPM) [10], regularized meshless method (RMM) [11,12], modified method of fundamental solutions (MMFS) [13], boundary collocation method (BCM) [14], and boundary distribution source method (BDS) [15], to name just a few. The key difference of these methods lies in the technique of desingularizing the source singularity of the fundamental solutions. The MFS, the most famous method of this group, regularizes the source singularity of fundamental solutions by placing an auxiliary boundary outside the physical domain. However, the MFS encounters two troublesome problems in the computation, *viz* ill-conditioning of the resulting interpolation matrices [16] and the choice of an optimal auxiliary boundary [17,18].

The SBM, a recent meshless method proposed by Chen et al. [8,19,20], replaces the source singularity with a desingularized value named Origin Intensity Factor (OIF), and therefore excludes the aforementioned auxiliary boundary associated with the MFS. Moreover, the SBM interpolation matrices are better conditioned. In this paper, we firstly derive the value of the OIF for the natural logarithm function, and thus obtain the OIFs of some fundamental solutions in 2D cases. The obtained analytical OIFs avoid the numerical implementation on the determination of the OIF, thereby improving the efficiency of the original SBM formulation. However, the SBM, akin to the BEM, is only applicable when the fundamental solutions of the governing operator are available.

It should be noted that the governing equation for heterogeneous heat conduction problems is a partial differential equation with variable-coefficients, whose response to a Dirac perturbation is nonsymmetric. As a consequence, it is mathematically impossible to derive the fundamental solutions for problems with arbitrary heat conductivities. Kassab [21] developed locally radially symmetric fundamental solutions with a generalized forcing function rather than a Dirac Delta function. In his generalized boundary integral equation, the influence of the domain residual, i.e. the generalized forcing function, is partially mitigated via an amplification factor. Their results show that the generalized boundary integral equation yields good results in a broad range of non-homogeneous problems [22]. Nevertheless in the meshless collocation methods, the domain residual is inevitable and may bring numerical instability [23,24]. On the other hand, Wang et al. [25] made use of the MFS coupled with the analog equation method to retrieve an arbitrary FGM problem as an equivalent homogeneous problem with a generalized body force. There are, however, collocation points in the domain to evaluate body force terms, which defeat the boundary only feature. Consequently, the primary challenge is to seek the fundamental solutions.

Some pioneering work had been undertaken by Abdullah, Clements and Ang [26] and Shaw [27] in developing the fundamental solutions for certain material property variations. Shaw and Manolis [28] constructed the fundamental solution for the heat conduction in a stochastic, heterogeneous medium where the thermal conductivity varies linearly along one direction and its slope consists of a constant plus a zero-mean random part. They also employed a conformal mapping technique to derive the fundamental solution of the heat conduction in 2D for those materials with the thermal conductivity varying as Bessel functions [29]. Gray and his cooperators considered the fundamental solution for both isotropic [30] and anisotropic [31] thermal conductivity varying exponentially with variable transformation. Kuo and Chen [32] also studied the transient heat transfer in the anisotropic solid with an exponentially grading along one direction. Marin and Lesnic [33] dealt with the nonlinear heat conduction problems in exponentially graded materials in favor of the Kirchhoff transformation. Sutradhar and Paulino [34] derived the fundamental solutions for heat conduction with the quadratic, exponential and trigonometric material variations of the thermal conductivity in one direction.

In this paper, the SBM is firstly formulated for the heat conduction problems with spatially varying conductivity. Three types of the thermal conductivity, *viz* quadratic, exponential or trigonometric material variations, are taken into consideration. The fundamental solutions of the governing equation are derived based on the variable transformation. The numerical results confirm the feasibility of the present SBM formulation and derived fundamental solutions.

The rest of the paper is constructed as follows: in Section 2, the governing equations and the derivation of its fundamental solutions are presented. In Section 3, the SBM for the steady-state heat conduction in heterogeneous material is formulated. In Section 4, several benchmark numerical examples are studied to illustrate the validity of the SBM. And finally, some conclusions are summarized in Section 5.

2. Governing equations and its fundamental solutions

The governing equation for steady state heat conduction problems in isotropic media with a spatially varying thermal conductivity can be expressed as

$$\nabla \cdot (k(\mathbf{x})\nabla T(\mathbf{x})) = 0 \tag{1}$$

with the boundary conditions

$$\begin{aligned} &T(\mathbf{x}) = h_D(\mathbf{x}), \quad \mathbf{x} \subset \Gamma_D, \\ &q(\mathbf{x}) = -k(\mathbf{x})\frac{\partial T(\mathbf{x})}{\partial \mathbf{n}} = h_N(\mathbf{x}), \quad \mathbf{x} \subset \Gamma_N \end{aligned} \tag{2}$$

where $k(\mathbf{x})$ is the spatially varying thermal conductivity, T is the temperature, q is the heat flux, $\boldsymbol{x}$ means the spatial coordinates, and $\boldsymbol{n}$ is the outward normal vector, $h_D(\mathbf{x})$ and $h_N(\mathbf{x})$ stand for measured data specified on the boundary, Ω is a bounded domain and $\partial\Omega = \Gamma_D \cup \Gamma_N$ denotes the whole physical boundary.