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Computing the outer and group inverses through elementary row operations

Jun Ji

Department of Mathematics and Statistics, Kennesaw State University, Kennesaw, GA 30144, United States

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ABSTRACT

A new method based on elementary row operations for computing the outer inverse of a given constant matrix is presented. When this method is applied to matrices of index one, a new expression for the group inverse is derived. Through this expression, a more efficient method for computing the group inverse of the square matrix A of order n with rank $r \leq 0.725n$ is developed.

1. Introduction

Throughout we shall use the notation of [1,2]. C^n and $C_r^{m\times n}$ stand for the n-dimensional complex vector space and the set of $m \times n$ matrices over complex field C with rank r, respectively. As always, the range, null space, and conjugate transpose of a matrix A will be denoted by $R(A)$, $N(A)$, and A^*, respectively. Let $A \in C_r^{m\times n}$, T be a subspace of C^n of dimension $s \leq r$ and S be a subspace of C^m of dimension $m - s$ such that

$$AT \oplus S = C^m. \tag{1}$$

Then there exists a unique matrix X such that $XAX = X$, $R(X) = T$, and $N(X) = S$. This matrix X is called the outer inverse, or $\{2\}$-inverse, of A with prescribed range T and null space S and denoted by $A_{T,S}^{(2)}$. This generalized inverse has many practical applications [1,3–5].

In addition, suppose $G \in C_s^{n\times m}$ satisfies

$$R(G) = T \quad \text{and} \quad N(G) = S \tag{2}$$

for the subspaces T and S in (1). Define $A^\sharp = N^{-1}A^*M$ for positive definite matrices $M \in C^{m\times m}$ and $N \in C^{n\times n}$. For the case when $m = n$, the smallest nonnegative integer l such that $\operatorname{rank}(A^l) = \operatorname{rank}(A^{l+1})$ is called the index of A and is denoted by $\operatorname{Ind}(A)$. It is well-known that

$$A_{T,S}^{(2)} = \begin{cases} A^\dagger & \text{if } G = A^*; \\ A_{M,N}^\dagger & \text{if } G = A^\sharp; \\ A^g & \text{if } m = n,\ G = A \text{ with } \operatorname{Ind}(A) = 1; \\ A^D & \text{if } m = n,\ G = A^l \text{ with } l \geq \operatorname{ind}(A); \\ A_{(L)}^{(-1)} & \text{if } R(G) = L \text{ and } N(G) = L^\perp; \\ A_{(L)}^{(\dagger)} & \text{if } R(G) = S \text{ and } N(G) = S^\perp. \end{cases}$$

E-mail address: jji@kennesaw.edu.

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Thus, the outer inverse $A^{(2)}_{T,S}$ provides a unified treatment for several generalized inverses, including the Moore–Penrose inverse $A^{\dagger}$, the weighted Moore–Penrose inverse $A^{\dagger}_{M,N}$, the group inverse A^{g}, the Drazin inverse A^{D}, the Bott–Duffin inverse $A^{(-1)}_{(L)}$, and the generalized Bott–Duffin inverse $A^{(\dagger)}_{(L)}$. The group inverse A^g is the special Drazin inverse when $\mathrm{Ind}(A) \leq 1$. For the case when A is a non-singular square matrix, we have $\mathrm{Ind}(A) = 0$ and $A^g = A^{\dagger} = A^{-1}$.

Many methods for the computation of $A^{(2)}_{T,S}$ have been proposed over the past thirty years. The iterative methods can be found in [6–10]. An algebraic perturbation method is given in [11]. The methods based on full-rank representation and QR decomposition are proposed in [12,13]. The Gauss–Jordan-like elimination methods for the outer inverse have been recently developed in the literature [14–16] which are direct extensions of those for the Moore–Penrose inverse [17–19] to the outer inverse.

The algorithm proposed by Sheng, Chen, and Gong [14] for computing the outer inverse $A^{(2)}_{T,S}$ starts from elementary row operations on $[GA \mid I]$ and the one of Sheng and Chen [16] begins with the elementary row and column operations on

$$\begin{bmatrix} GAG & G \\ G & 0 \end{bmatrix}. \tag{3}$$

While these methods provide ways of computing $A^{(2)}_{T,S}$, both involve the construction of auxiliary matrices containing GA before elementary transformations can be applied. Therefore, not only these methods increase the computational costs by constructing either GA as in [14] or GAG as in [16] but also they worsen the condition number of the problem. The method in [15] starts with the elementary row operations on $[G \mid I]$. This approach is close to the one in [14] but is more stable and faster as it does not directly involve the computation of GA.

The classical Gauss–Jordan elimination for the inverse of a nonsingular matrix is the method of choice for small problems since it can be easily carried out "by hand". All the afore-mentioned methods of elementary operations for the outer inverse have the same advantage. But we need to be aware of the fact that these methods of elementary operations are not reduced to the classical Gauss–Jordan elimination method for the regular inverse when applied to a nonsingular matrix since in such a case, the choice for G is A^* and none of them starts with the elementary row operations on $[A \mid I]$ as the classical counterpart does. A method using Gauss–Jordan elimination starts with $[A \mid I]$ for the Moore–Penrose inverse was recently developed in [20]. In this paper, we will propose an alternative method of elementary row operations for outer inverse by performing row operations first on $[G^* \mid I]$, following the lines of [20]. Our approach is more like the one in [16] by working with a bordered matrix and the outer inverse is read off from the computed result but there is no need for forming GAG. Moreover, the new approach will lead us to a new expression for the group inverse when applied to matrices of index one. Based on this expression, a more efficient method for computing the group inverse of matrices with $r \leq 0.725n$ is developed.

2. The new method for the outer inverses

We start our discussion by applying elementary row operations on $[G^* \mid I]$ and this results in $[B \mid P]$ where B is in the reduced row echelon form. Let P be the product of all the elementary matrices representing these elementary row operations. We can write

$$P[G^* \mid I] = [PG^* \mid P] = [B \mid P].$$

Denote the matrix B by

$$B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \quad \text{where } B_1 \in C_s^{s\times n}.$$

Next, we apply elementary row operations on $[B^* \mid I]$. Due to the fact that B is in the reduced row echelon form, B^* is transformed by elementary row operations to

$$C = \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix} \tag{4}$$

or equivalently there exists a nonsingular matrix $Q \in C^{n\times n}$ such that

$$Q^*[B^* \mid I] = [C \mid Q^*].$$

Therefore, we have

$$PG^* = B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \quad \text{and} \quad Q^*B^* = C = \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}. \tag{5}$$

Theorem 1. *Let $A \in C_r^{m\times n}$, T be a subspace of C^n of dimension $s \leq r$ and S be a subspace of C^m of dimension $m - s$ such that (1) is satisfied. Assume that $G \in C_s^{n\times m}$ satisfies (2). P and Q are two nonsingular matrices such that (5) is satisfied with*