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Axioms of adaptivity

C. Carstensen [a,b], M. Feischl [c,*], M. Page [c], D. Praetorius [c]

[a] Institut für Mathematik, Humboldt Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany

[b] Department of Computational Science and Engineering, Yonsei University, 120-749 Seoul, Republic of Korea

[c] Institute for Analysis and Scientific Computing, Vienna University of Technology, Wiedner Hauptstraße 8-10, A-1040 Wien, Austria

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ABSTRACT

This paper aims first at a simultaneous axiomatic presentation of the proof of optimal convergence rates for adaptive finite element methods and second at some refinements of particular questions like the avoidance of (discrete) lower bounds, inexact solvers, inhomogeneous boundary data, or the use of equivalent error estimators. Solely four axioms guarantee the optimality in terms of the error estimators.

Compared to the state of the art in the temporary literature, the improvements of this article can be summarized as follows: First, a general framework is presented which covers the existing literature on optimality of adaptive schemes. The abstract analysis covers linear as well as nonlinear problems and is independent of the underlying finite element or boundary element method. Second, efficiency of the error estimator is neither needed to prove convergence nor quasi-optimal convergence behavior of the error estimator. In this paper, efficiency exclusively characterizes the approximation classes involved in terms of the best-approximation error and data resolution and so the upper bound on the optimal marking parameters does not depend on the efficiency constant. Third, some general quasi-Galerkin orthogonality is not only sufficient, but also necessary for the R-linear convergence of the error estimator, which is a fundamental ingredient in the current quasi-optimality analysis due to Stevenson 2007. Finally, the general analysis allows for equivalent error estimators and inexact solvers as well as different non-homogeneous and mixed boundary conditions.

1. Introduction & outline

1.1. State of the art

The impact of adaptive mesh-refinement in computational partial differential equations (PDEs) cannot be overestimated. Several books in the area provide sufficient evidence of the success in many practical applications in computational sciences and engineering. Related books from the mathematical literature, e.g., [1–5] provide many a posteriori error estimators which compete in [6,7], and overview articles [8–10] outline an abstract framework for their derivation.

This article contributes to the theory of optimality of adaptive algorithms in the spirit of [11–18] for conforming finite element methods (FEMs) and exploits the overall mathematics for nonstandard FEMs like nonconforming methods [19–27]

* Corresponding author.
E-mail addresses: cc@math.hu-berlin.de (C. Carstensen), michael.feischl@tuwien.ac.at (M. Feischl), marcus.page@tuwien.ac.at (M. Page), dirk.praetorius@tuwien.ac.at (D. Praetorius).

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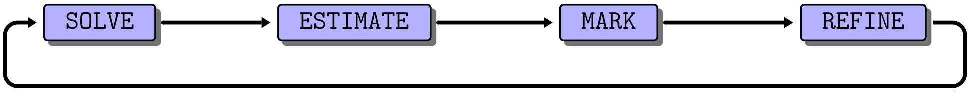
and mixed formulations [28–31] as well as boundary element methods (BEMs) [32–36] and possibly non-homogeneous or mixed boundary conditions [37–39].

Four main arguments compose the set of axioms and identify necessary conditions for optimal convergence of adaptive mesh-refining algorithms. This abstract framework answers questions like: What is the state-of-the-art technique for the design of an optimal adaptive mesh-refining strategy, and which ingredients are really necessary to guarantee quasi-optimal rates? The overall format of the adaptive algorithm follows the standard loop



in the spirit of the pioneering works [40,41]. This is the most popular version of adaptive FEM and BEM in practice. While earlier works [42–44] which faced an abstract framework for adaptivity were only concerned with convergence of adaptive conforming FEM, the present article provides a problem and discretization independent framework for convergence and quasi-optimal rates of adaptive algorithms. In particular, this includes adaptive FEM and BEM with conforming, nonconforming, as well as mixed methods.

1.2. Contributions of this work

The contributions in this paper have the flavor of a survey and a general description in the first half comprising Sections 2–6, although the strategy is different from the main stream of, e.g., [14–17] and the overview articles like [45,46]: The efficiency is not used and data approximation terms do no enter in the beginning. Instead, the optimality is firstly proved in terms of the a posteriori error estimators. This approach of [18,39] appears natural as the algorithm only concerns the estimator rather than the unknown error. Efficiency solely enters in a second step, where this first notion of optimality is shown to be equivalent to optimality in terms of nonlinear approximation classes which include best approximation error plus data approximation terms [15]. In our opinion, this strategy enjoys the following advantages (a)–(b):

(a) Unlike [14–17], the upper bound for adaptivity parameters which guarantee quasi-optimal convergence rates, is independent of the efficiency constant. Such an observation might be a first step to the mathematical understanding of the empirical observation that each adaptivity parameter $0 < \theta \leq 0.5$ yields optimal convergence rates in the asymptotic regime.

(b) Besides boundary element methods, see e.g. [32,33,47,48], there might be other (nonlinear) problems, where an optimal efficiency estimate is unknown or cannot even be expected. Then, our approach guarantees at least that the adaptive strategy will lead to the best possible convergence behavior with respect to the computationally available a posteriori error estimator.

The first half of this paper discusses a small set of rather general axioms (A1)–(A4) and therefore involves several simplifying restrictions such as an exact solver. Although the axioms are motivated from the literature on adaptive FEM for linear problems and constitute the main ingredients for any optimality proof in literature so far, we are able to show that this minimal set of four axioms is sufficient to prove optimality. Moreover, linear convergence of the scheme is even characterized in terms of a novel quasi-orthogonality axiom (see Section 4.4). Finally, optimality of the marking criterion is essentially equivalent to the discrete reliability axiom (see Section 4.5). Therefore, two of these four axioms even turn out to be necessary. Unlike the overview articles [45,46], the analysis is not bound to a particular model problem, but applies to any problem within the framework of Section 2 and therefore sheds new light onto the theory of adaptive algorithms. In Section 5, these axioms are met for different formulations of the Poisson model problem and allow to reproduce and even improve the state-of-the-art results from the literature for conforming AFEM [14,15], nonconforming AFEM [20,22,25,28], mixed AFEM [19,29,31], and ABEM for weakly-singular [32–35] and hyper-singular integral equations [33,36]. Moreover, further examples from Section 6 show that our frame also covers conforming AFEM for non-symmetric problems [17,18,49], linear elasticity [30,50,51], and different formulations of the Stokes problem [50–55]. We thus provide a general framework of four axioms that unifies the diversity of the quasi-optimality analysis from the literature. Given any adaptive scheme that fits into the above frame, the validity of those four axioms guarantee optimal convergence behavior independently of the concrete setup.

To illustrate the extensions and applicability of our axioms of adaptivity (A1)–(A4), the second half of this paper treats further advanced topics and contributes with new mathematical insight in the striking performance of adaptive schemes.

First, Section 7 generalizes [21] and analyzes the influence of inexact solvers, which are important for iterative solvers, especially for nonlinear problems. This also gives a mathematically satisfactory explanation of the stability of adaptive schemes against computational noise as e.g. rounding errors in computer arithmetics.

Second, the historic development of adaptive algorithms focused on residual-based a posteriori error estimators, but all kinds of locally equivalent a posteriori error estimators can be exploited as refinement indicators as well. Section 8 provides the means to show optimal convergence behavior even in this case and extends [16] which is restricted to a patch-wise marking strategy with unnecessary refinements. The refined analysis in this paper is essentially based on a novel equivalent mesh-size function. It provides a mathematical background for the standard AFEM algorithm with facet-based and/or non-residual error estimators. To illustrate the analysis from Section 8, Section 9 provides several examples with facet-based