Computers and Mathematics with Applications 67 (2014) 1293–1308

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Stabilized explicit Runge–Kutta methods for multi-asset American options

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ARTICLE INFO

Article history:
Received 21 November 2012
Received in revised form 17 January 2014
Accepted 24 January 2014

Keywords:
Multi-asset options
Stabilized explicit Runge–Kutta methods
American options
Non-smooth payoffs

ABSTRACT

American derivatives have become very popular instruments in financial markets. However, they are more complicated to price than European options since at each time level we have to determine not only the option value but also whether or not it should be exercised. Several procedures have been proposed to dissolve these difficulties, but they usually involve the solution of nonlinear partial differential equations (PDEs). In the case of multi-dimensional problems, solving these equations is a very challenging task.

In this paper we propose Stabilized Explicit Runge–Kutta (SERK) methods to solve this kind of problems. They can easily be applied to many different classes of problems with large dimensions and they have low memory demand. Since these methods are explicit, they do not require algebra routines to solve large nonlinear systems associated to ODEs (as, for example, LAPACK and BLAS packages or multigrid or iterative methods applied together with Newton-type algorithms) and are especially well-suited for the method of lines (MOL) discretizations of parabolic PDEs.

1. Introduction

Since the 1970s the well-known Black–Scholes formula derived by Black and Scholes [1] and Merton [2] computes the value of a European option based on the underlying asset, the strike price, the volatility of the asset and the time until expiry. However, a European option can be exercised only at the expiry date whereas an American option has the additional feature that exercise is permitted at any time during the life of the option. American derivatives are getting popular trading instruments in present-day financial markets. On the other hand, pricing an American option is more complicated since at each time step we have to determine not only the option value but also whether or not it should be exercised. Hence, this turns the valuation of an American option into a free boundary problem.

Typically, at each time level, there is a particular value of the asset which marks the boundary between two regions: on the one side, one should hold the option and on the other side, one should exercise it. Assuming that investors act optimally, the value of an American option cannot fall below the value that would be obtained if it was exercised early. Effectively, there are different ways to study American options, two common ideas are transforming the original linear European pricing equation into a (i) nonlinear Partial Differential Equation (PDE) adding a nonlinear penalty source term or (ii) a linear complementary problem similar to the one proposed later in Section 2.

In this paper we consider mostly American options where the payoff depends on several underlying assets. Such option prices can be modeled by higher dimensional generalizations of the original Black–Scholes equation. Various numerical

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http://dx.doi.org/10.1016/j.camwa.2014.01.018
0898-1221/

techniques can be applied to price multi-variate derivatives (see [3,4]). Higher dimensional generalizations of lattice binomial methods can be used (see [5], where European options based on three underlying options are solved numerically). Another way of pricing multi-asset derivatives is by using a Monte Carlo simulation technique [6–8]. In a wide range of scientific fields, including the numerical valuation of financial derivatives, finite element and finite difference methods (FEM and FDM) are popular, see [9–13]. All these schemes lead to a very CPU demanding procedure, especially when the PDE is nonlinear, in which case they require algebra routines to solve very large systems of nonlinear equations.

Traditionally classical explicit methods have not been used for these problems due to their stability limitations. However, in this paper, we propose the use of Stabilized Explicit Runge–Kutta (SERK) methods to solve this kind of problems. These schemes are explicit methods with extended stability domains, usually along the negative real axis. They require more matrix–vector multiplications in each step than a traditional explicit method, two of them are required to obtain second-order convergence and the rest are used to quadratically extend the region where they are stable. Thus, they can easily be applied to large problem classes, they have low memory demand and are especially suited for discretizations using the method of lines (MOL) of two and three dimensional parabolic PDEs.

Additionally, these algorithms allow adaptation of the time step with practically no extra cost. The solution of the PDE and its derivatives are not totally smooth after the initial or an intermediate point of the interval of integration (due to the condition $V(x, T) = \max(x - K, 0)$). Therefore, when it is necessary to obtain a chosen error, the adaptation of the mesh reduces the number of matrix–vector multiplications (see numerical results at Example 1.a). Hence, they are very suitable for American options.

Other stabilized explicit Runge–Kutta methods have been derived previously. In [14] the authors proposed a new second-order algorithm whose length of the stability interval is $0.8s^2$ (instead of $0.65s^2$ as in [15], s being the number of stages of the explicit algorithm) and with better stability properties than DUMKA or ROCK methods [16,17], which have some difficulties when any of the eigenvalues of the Jacobian are very large. This paper focuses on the study of SERK methods applied to European, but mostly American options with non-smooth payoffs.

This paper is organized as follows. In Section 2 the model problems are presented. In Section 3 we describe the procedure to obtain polynomials with large stability regions and derive Runge–Kutta methods with these polynomials as stability functions. These methods have been implemented in a new variable step and number of stages code called SERK2v2 presented in Section 4. Finally, Section 5 contains numerical experiments for various options. First, two simple examples in one dimension are considered to compare the new methods with other well-known methods. Later we show the efficiency for higher dimensional problems.

2. Pricing American options

Black, Scholes and Merton derived a model equation to compute the price V of any contingent claim written on a stock. To obtain a parabolic problem, a simple transformation is used to get the following forward PDE where all eigenvalues have a negative real part. After this transformation the t variable is the time to expiry:

$$\frac{\partial V(x,t)}{\partial t} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V(x,t)}{\partial x^2} + (r-D)x\frac{\partial V(x,t)}{\partial x} - rV(x,t), \tag{1}$$

where x is the price of the stock, r is the annualized risk-free interest rate, σ is the volatility of the stock's returns and D is the dividend yield. The parameters r, σ and D are given constants.

For pricing of an American call option the Black–Scholes model described above takes the form of a moving boundary problem and it is necessary to add the following conditions (see [18,19])

$$\begin{aligned}
&V(x,0) = \max(x-K,0), \quad x \ge 0,\\
&\frac{\partial V}{\partial x}(\bar{S},t) = 1,\\
&V(\bar{S},t) = \bar{S} - K,\\
&V(0,t) = 0,\\
&\bar{S}(0) = \max\left(K, \frac{rK}{D}\right),
\end{aligned}$$

where $\bar{S}(t)$ represents the free (and moving) boundary. Since for an American option early exercise is permitted, the value $V(x, t)$ of the option must satisfy the positivity constraint $V(x,t) \ge \max(x-K,0),\ x \ge 0,\ 0 \le t \le T$.

MOL is the most popular method for solving PDEs in engineering. The idea is to first reduce a time-dependent PDE into a system of ordinary differential equations (ODEs) in time via semi-discretization in space. Then this system of ODEs in time can further be solved efficiently by many well-developed ODE solvers.

There are many ways to discretize (1). Usually central difference approximations are used in both, first and second order derivatives with respect to the asset price. Then we obtain

$$\frac{dV_i}{dt} = \frac{1}{2}\sigma^2 x_i^2 \frac{V_{i+1} - 2V_i + V_{i-1}}{\Delta x^2} + (r-D)x_i\frac{V_{i+1} - V_{i-1}}{2\Delta x} - rV_i, \quad i = 1, 2, \ldots, Nx_\infty - 1, \tag{2}$$

where x_∞ is the end point of the integration interval (we will consider that Nx_∞ is an integer).