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A sparse mesh for Compact Finite Difference—Fourier solvers with radius-dependent spectral resolution in circular domains

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ABSTRACT

This paper presents a new method for the resolution of elliptic and parabolic equations in circular domains. It can be trivially extended to cylindrical domains. The algorithm uses a mixed Fourier-Compact Finite Difference method. The main advantage of the method is achieved by a new concept of mesh. The topology of the new grid keeps constant the aspect ratio of the cells, avoiding the typical clustering for radial structured meshes at the center. The reduction of the number of nodes has as a consequence the reduction in memory consumption. In the case of fluid mechanics problems, this technique also increases the time step for a constant Courant number. Several examples are given in the paper which show the potential of the method.

1. Introduction

The resolution of non-linear parabolic and elliptic equations in circular domains is of great interest for several branches of knowledge. In particular, the understanding of the kinematics and the dynamics of turbulent flows in pipes remains one of the challenges for the next decade. As an example, 50% of the energy losses in large pipes are originated in the first millimeters from the wall [1–3]. Spectral or spectral-like methods are frequently chosen to solve these sort of problems, due to both their great precision and their high ratio of mesh size to computational cost. These methods, when applied on circular domains, are usually formulated in polar coordinates, with the main advantage that the boundary conditions can be imposed in a straightforward manner. Nevertheless, there are two particularities to be dealt with. On one hand, the origin is a pole and it needs a special treatment, which usually involves the use of artificial boundary conditions. On the other hand, for structured radial grids there is a mesh size reduction at the center. For instance, when dealing with turbulent flows in pipes, the smaller structures, which define the mesh structure, are close to the wall, whereas the larger ones are at the centerline of the pipe, thus giving rise to an unnecessary increase in memory requirements. Additionally, the Courant–Friedrichs–Lewy condition (CFL from now on) would impose a very short time step.

The problem at the origin has been addressed in different ways. Chen et al. [4] used spectral collocation methods in order to increase the order of the equation. Li et al. [5] simulated the Navier–Stokes equations with three Chebyshev–Fourier spectral collocation methods. Pure spectral methods have been also proposed, as the Fourier–Legendre discretization used by Z. Qiu et al. [6], and the Fourier-generic orthogonal polynomials used by Matsushima and Marcus [7]. A coordinate system transformation is another possible technique. Heinrichs [8] used conformal mapping to transform a Cartesian coordinate system into a polar system, and a similar approach was proposed by Hansen et al. [9]. The transformation was applied to

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the nodes closest to the center, and the components, already in Cartesian form, were then averaged, giving as a result the value of the pole.

The accumulation of grid points at the center of the domain can also be compensated. Kwan [10] proposed a spectral-Galerkin method with a quadratic transform in the radial direction to improve the clustering at the center. Akselvoll and Moin [11] solved the Navier–Stokes equations by dividing the circular domain into two separate regions, the core and the outer region, respectively. At the core region some of the terms were treated explicitly, whereas at the outer region all the terms were solved implicitly. They also used different time schemes for each region to improve the performance without affecting the accuracy.

An additional possibility is the use of a Fourier-Compact Finite Difference (CFD) discretization developed by Lee [12], as that proposed by Lai [13], who imposed symmetry and antisymmetry conditions at the center (asymptotic behavior) applied on phantom points for odd and even values of each wave number.

In the present paper a new algorithm is presented. It uses a sixth-order CFD method in the radial direction, and spectral Fourier decomposition in the azimuthal one. When applied to an elliptic equation, the algorithm produces a set of one-dimensional radial equations. Each equation set is represented by a compact sixth-order finite difference discretization. As a consequence, the error is bounded by the maximum wave number, and by the radial spacing. The aim of this work is to optimize the CFD-Fourier refinement to solve the system as accurately as possible, minimizing the computational resources required. This objective is achieved by the creation of a structured radial mesh with thresholded aspect-ratio and compatible with the Fourier decomposition in the azimuthal coordinate. The main characteristic of the grid is the reduction of the number of computed wave lengths for the internal radii, so that the memory storage is greatly reduced and the computational time is decreased by some orders of magnitude.

The paper is organized as follows. Section 2 is devoted to explain the numerical method and the meshing techniques. The mesh is described in Section 3 and some illustrative examples are given in Section 4. Finally, in Section 5 the main conclusions of the work are pointed out.

2. Numerical method

2.1. The non-linear elliptic equation

The general two-dimensional non-linear partial differential equation considered in this work is

$$\partial_t \vec{u} + N\left(\vec{u}\right) - \frac{1}{\beta} L\left(\vec{u}\right) = \vec{S}, \tag{1}$$

solved in a circular domain of coordinates r and ϕ. In Eq. (1) L and N stand for the linear and the non-linear term, respectively, while $\vec{S}$ is a source term that may depend on t, r and θ. β is a positive constant whose finality is to weight the diffusion speed. The smaller it is, the faster is the diffusion. In the case of the Navier–Stokes equations for incompressible flows, it is the Reynolds number. As an example, which will be discussed later on, for the classical heat equation $N(\vec{u}) = 0, L(\vec{u}) = \nabla^2\vec{u}$ and β is the inverse of the thermal diffusivity, $\alpha = 1/\beta$. This equation is solved in a circular domain with unit radius, so the polar coordinate system (r, θ) is the most appropriate one. The domain is then delimited to $\Omega : [0, 1] \times [0, 2\pi)$.

It is also worth mentioning that in the case of the Navier–Stokes equations, if the domain is periodic along the pipe length the use of a Fourier spectral method reduces the problem to the solution of a (probably large) set of problems like (1).

2.2. General aspects

The Runge–Kutta methods [14] are the time schemes most widely used for fluid dynamics computations. This is due to its high accuracy even when relatively long time steps are used. However, for the sake of simplicity, an Euler implicit method has been chosen here, the generalization to more complicated schemes being straightforward. While in the azimuthal coordinate an obvious choice is the use of a Fourier decomposition method, in the radial direction a CFD scheme [12] has been used, due to its flexibility in choosing the points and to its spectral-like resolution.

2.3. Space discretization: CFD-Fourier

Applying a Fourier spectral expansion along θ, a set of one-dimensional ordinary differential equations is obtained. Let n be the number of points in the azimuthal coordinate. This also is the number of Fourier modes. The equation for each wavenumber κ is

$$\partial_t \hat{\vec{u}}_\kappa + N_\kappa\left(\hat{\vec{u}}_\kappa\right) - \frac{1}{\beta} L_\kappa\left(\hat{\vec{u}}_\kappa\right) = \hat{\vec{S}}_\kappa, \tag{2}$$

where the $\widehat{\ }$ symbol means that the variable is expressed in the wavenumber domain. Both the linear and the non-linear operators depend on the wave number; however, a part of L remains constant. Then, it is convenient to split L into this constant term L_C and the κ-dependent one L_r^κ, as

$$L_\kappa = L_C + L_r^\kappa. \tag{3}$$