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A mixed multiscale finite element method for convex optimal control problems with oscillating coefficients

Yanping Chen [a,*], Yunqing Huang [b,c], Wenbin Liu [d], Ningning Yan [e]

[a] School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China

[b] Hunan Key Laboratory for Computation and Simulation in Science and Engineering, Xiangtan University, Xiangtan 411105, Hunan, China

[c] School of Mathematics and Computational Science, Xiangtan University, Xiangtan 411105, Hunan, China

[d] KBS & IMS, University of Kent, Canterbury CT2 7NF, UK

[e] Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, China

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ABSTRACT

We study numerical approximation of convex optimal control problems governed by elliptic partial differential equations with oscillating coefficients. Since the objective functional contains flux, we approximate the problems using the mixed finite element methods. We first analyze the standard mixed finite element approximation schemes. Then, motivated by the numerical simulation of the primal variable and the flux in highly heterogeneous porous media, we use a multiscale mixed finite element method to solve the state equations. The multiscale finite element bases are constructed by locally solving Neumann boundary value problems. The analysis of the approximate control problems is carried out under the assumption that the oscillating coefficients are locally periodic, which allows us to use homogenization theory to obtain the asymptotic structure of the solutions, although the numerical schemes are designed for the general case.

1. Introduction

In this paper, we consider solving a class of two-dimensional, optimal control problems governed by elliptic partial differential equations with highly oscillatory coefficients. As the state equation, such general elliptic problems with separable two-scale coefficients often arise in composite materials, flows in porous media, and many other applications areas. Optimal control and optimization problems also arise in these connections. In practice, the oscillatory coefficients may contain many scales spanning over a great extent.

Optimal control problems are very important models in science and engineering numerical simulation. Efficient numerical methods are essential to successful applications of optimal control. Over the past decades, the finite element method seems to be the most widely used numerical method in computing optimal control problems. There have been extensive theoretical studies for finite element approximation of various optimal control problems. A comprehensive review of the finite element method for partial differential equations and optimal control problems can be found in, for example, [1–3], and the references therein. For optimal control problems governed by linear elliptic or parabolic state equations, a priori error estimates of finite element approximation were established long ago, see, for example, [4,5,1,3]. Furthermore a priori error estimates have been also established for some important flow control problems, see, e.g., [6]. A priori error estimates have

* Corresponding author.
E-mail addresses: yanpingchen@scnu.edu.cn (Y. Chen), huangyq@xtu.edu.cn (Y. Huang), W.B.Liu@ukc.ac.uk (W. Liu), ynn@amss.ac.cn (N. Yan).

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also been obtained for a class of state constrained control problems in [7,8], though the state equation is assumed to be linear. In [9], the linear assumption has been removed by reformulating the control problem as an abstract optimization problem in some Banach spaces and then applying non-smooth analysis. In fact, the state equation there can be a variational inequality.

A posteriori error estimates for optimal control problems with control constraints were established in, for example, [10–12,2]. The results have been found very useful in developing adaptive finite element approximation to the control problems, which can save a great deal of computational work, see [13] for instance. Recently, an adaptive finite element method for the estimation of distributed parameter in elliptic equation was discussed in [14]. Adaptive finite element methods for optimal control problems with state constraints were considered in, for example, [15,16].

Optimal control problems by elliptic partial differential equations with oscillating coefficients have become one of challenging research fields in control theory, owing to its major applications in the flow control. More recently, we examine an important class of finite element approximation for convex optimal control problems governed by elliptic partial differential equations with oscillating coefficients. Those belong to the so called multi-scale problems. We note that standard finite element methods do not yield good numerical approximations when the mesh size $h > \varepsilon$. To overcome this, an important numerical methods have been proposed for solving the multi-scale problems, such as the multiscale finite element methods. There exists an extensive body of literature; for example, see [17–19]. Such problems occur in applications involving control of water injection into oil-wells, and properties design of composite materials. Our recent work (see [20]) serves as a contribution to the development of the multiscale finite element method for solving optimal control problems by elliptic partial differential equations with oscillating coefficients.

In finite element method, mixed finite elements are widely used to approximate flux variables, see [21]. The local conservation of velocity flux is an important property in the mixed finite element methods. The violation of this local conservation property will lead to leakage of velocity flux. This will deteriorate the accuracy of the numerical solution for long-time computations. This is the reason why mixed finite element methods are attractive for porous medium simulations. To our best knowledge, there is only very limited research work on analyzing such elements for optimal control. In recent years, we have investigated a mixed finite element method for optimal control problems, whose objective functional contains both flux and state variables. We have carried out carefully some research on superconvergence properties, a priori error estimates, and a posteriori error estimates to validate our mixed finite element methods for optimal control problems and high-order accuracy, spectral methods for some flow control problems; see [22–25]. Thus it is worth to investigate mixed finite element methods for the control problems involving flux control. The motivation of this paper is to explore the possibility of applying mixed multiscale finite element methods to numerical computation of optimal control problems by elliptic partial differential equations with oscillating coefficients, where we carry out a further investigation of a type of elliptic linear optimal control problems with flux control.

We consider the following convex optimal control problems governed by elliptic partial differential equations with oscillating coefficients:

$$\min_{u\in K\subset L^2(\Omega_U)} \{g_1(\boldsymbol{p}) + g_2(y) + h(u)\}, \tag{1.1}$$

$$\nabla\cdot\boldsymbol{p} = f + Bu \quad \text{in } \Omega, \tag{1.2}$$

$$\boldsymbol{p} = -a\left(x, \frac{x}{\varepsilon}\right)\nabla y, \quad \text{in } \Omega, \tag{1.3}$$

$$y = 0, \quad \text{on } \partial\Omega, \tag{1.4}$$

where the bounded open set $\Omega \subset \mathbb{R}^2$, is a convex polygon or has a smooth boundary $\partial\Omega$, Ω_U is a bounded open set in $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega_U$, and K is a closed convex set in $L^2(\Omega_U)$. In the context of porous flow, Eqs. (1.2)–(1.4) are the pressure equations for two phase flow through a porous medium. Correspondingly, a is the ratio of the permeability and the fluid viscosity, and y represents the pressure. The velocity field is related to the pressure through Darcy's law (1.3). Here, g_1, g_2 and h are convex functionals. The coefficient matrix $a\left(x, \frac{x}{\varepsilon}\right) = a_{ij}\left(x, \frac{x}{\varepsilon}\right) \in L^\infty(\Omega; \mathbb{R}^{2\times 2})$ is a symmetric matrix which satisfies the uniform ellipticity condition:

$$\gamma|\boldsymbol{\xi}|^2 \le a_{ij}(x,\tilde{x})\xi_i\xi_j \le \gamma^{-1}|\boldsymbol{\xi}|^2 \quad \forall\,\boldsymbol{\xi}\in\mathbb{R}^2,\ x\in\overline{\Omega},\ \tilde{x}\in\mathbb{R}^2, \tag{1.5}$$

for some positive constant γ. Here, $f \in L^2(\Omega)$ and B is a continuous linear operator from $L^2(\Omega_U)$ to $L^2(\Omega)$.

The plan of this paper is as follows: In Section 2, we formulate mixed finite element approximation schemes for the optimal control problem. We study both the standard and the multiscale mixed finite element scheme for comparison. In Section 3, we present error analysis of the standard mixed finite element approximation for the optimal control, as the standard mixed scheme has not been properly analyzed yet for the control problem. We are then able to show the key weakness of the standard mixed scheme for the multiscale control problem, and also this section paves the way for analyzing the multiscale scheme. In Section 4, we further carry out convergence analysis for the multiscale mixed finite element scheme.

In this paper we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|\phi\|_{m,p}^p = \sum_{|\alpha|\le m}\|D^\alpha\phi\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega) = \{\phi\in W^{m,p}(\Omega) : \phi|_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, $\|\cdot\|_m = \|\cdot\|_{m,2}$ and $\|\cdot\| = \|\cdot\|_{0,2}$. In addition C denotes a generic positive constant independent of h and ε, and C_ε a generic positive constant independent of h.

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