Computers and Mathematics with Applications 70 (2015) 353–458

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Orientation embedded high order shape functions for the exact sequence elements of all shapes

Federico Fuentes *, Brendan Keith, Leszek Demkowicz, Sriram Nagaraj

The University of Texas at Austin, Institute for Computational Engineering and Sciences (ICES), 201 East 24th St, Austin, TX, 78712, USA

ARTICLE INFO

Article history:
Received 23 April 2015
Accepted 23 April 2015
Available online 28 June 2015
Communicated by N. Nigam and J. Schoeberl

Keywords:
High order shape functions
Orientation embedding
Energy spaces
Pyramid
Hierarchical shape functions
Hybrid meshes

ABSTRACT

A unified construction of high order shape functions is given for all four classical energy spaces (H^1, $H(\text{curl})$, $H(\text{div})$ and L^2) and for elements of "all" shapes (segment, quadrilateral, triangle, hexahedron, tetrahedron, triangular prism and pyramid). The discrete spaces spanned by the shape functions satisfy the commuting exact sequence property for each element. The shape functions are conforming, hierarchical and compatible with other neighboring elements across shared boundaries so they may be used in hybrid meshes. Expressions for the shape functions are given in coordinate free format in terms of the relevant affine coordinates of each element shape. The polynomial order is allowed to differ for each separate topological entity (vertex, edge, face or interior) in the mesh, so the shape functions can be used to implement local p adaptive finite element methods. Each topological entity may have its own orientation, and the shape functions can have that orientation embedded by a simple permutation of arguments.

1. Preliminaries

1.1. Introduction

In the context of finite elements, construction of higher order shape functions for elements forming the exact sequence has been a long standing activity in both engineering and numerical analysis communities. A comprehensive review of the subject can be found for example in [1,2] and references therein.

This document presents a self-contained systematic theory for the construction of a particular set of hierarchical, orientation embedded, H^1, $H(\text{curl})$, $H(\text{div})$, and L^2 conforming shape functions for elements of "all shapes", forming the 1D, 2D, and 3D commuting exact sequences discussed within. By elements of "all shapes", we specifically mean the segment (unit interval) in 1D, the quadrilateral and triangle in 2D, and the hexahedron, tetrahedron, prism (wedge) and pyramid in 3D (see Fig. 1.1).

There are many ways to construct sets of shape functions satisfying the aforementioned properties. However, we believe that in this work we have constructed a set which strikes an uncommon balance between simplicity and applicability. For all elements, and each associated energy space, we rely upon a simple methodology and a very small collection of ancillary functions to generate all of our shape functions. Furthermore, we have supplemented this text with a package written in Fortran 90 defining each function presented in this work.[1] For these reasons, when reproducing our work in their own

* Corresponding author.
E-mail address: federico@ices.utexas.edu (F. Fuentes).

[1] See the ESEAS library available at https://github.com/libESEAS/ESEAS.

http://dx.doi.org/10.1016/j.camwa.2015.04.027
0898-1221/

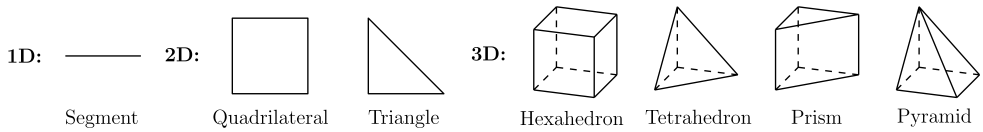

Fig. 1.1. Elements of "all shapes".

software, the readers should find the burden of implementation minimal. We hope that our exposition will be clear and useful, particularly to those less familiar with the subject.

For those at the forefront of shape function construction, we hope that our work will be intriguing if only for the elegance of our construction. Particularly, we evidence Section 9 on pyramid shape functions. The higher order discrete commuting exact sequence for this element appeared only recently in the work of Nigam and Phillips [3]. Our construction for the pyramid presents shape functions spanning each of their discrete energy spaces while maintaining compatibility with the other 3D elements. We also remark that, for any given mesh, our shape functions are fully compatible across adjacent interelement boundaries due to considering so-called orientation embeddings. Hence, no alterations of the shape functions are necessary at the finite element assembly procedure. Moreover, these orientation embeddings are handled almost effortlessly by simply permuting the entries of a few relevant functions.

With regard to the choice of geometry (shape and size) of master elements, we followed Demkowicz [1]. However, one of the key points is that our construction naturally applies to any other choice of master element geometries. Other specific choices we made when enumerating element vertices, edges, and faces, can also be modified with little effort to the preferences of the reader.

For completeness, we have chosen to thoroughly verify the mathematical properties and to give a sound geometrical interpretation of our constructions rather than only give the necessary shape functions to the reader. We concede that due to the depth of our presentation, and the expanse of our coverage, our offense lies only in the length of this document. However, in a sense, an abridged version of this work is already present in a set of tables summarizing all the shape functions. These can be conveniently consulted in Appendix E.

1.2. Energy spaces and exact sequences

Let $\Omega \subseteq \mathbb{R}^N$, with $N = 1, 2, 3$ be a domain. One arrives naturally at the *energy spaces* $H^1(\Omega)$, $H(\text{curl}, \Omega)$, $H(\text{div}, \Omega)$ and $L^2(\Omega)$ in context of various variational formulations, see e.g. Chapter 1 in [2] and [4,5]. Along with operations of gradient, curl and divergence (understood in the sense of distributions), these spaces form the so-called *complexes*, i.e. the composition of any two operators in the sequence reduces to the trivial operator. The 1D complex, where $\Omega \subseteq \mathbb{R}$, provides the simplest example:

$$\mathbb{R} \xrightarrow{\text{id}} H^1(\Omega) \xrightarrow{\nabla} L^2(\Omega) \xrightarrow{0} \{0\}.$$

Here, the symbol $\mathbb{R}$ denotes constant functions, and $\{0\}$ is the trivial vector space consisting of the zero function only. By using the name of *complex*, we communicate two simple facts: (a) the derivative of a constant function is zero, and (b) the composition of derivative (in fact, any linear operator) with the trivial (zero) operator is trivial as well. Equivalently, we can express the same facts by using null spaces and ranges of the involved operators:

$$\mathsf{R}(\text{id}) \subseteq \mathsf{N}(\nabla) \quad \text{and} \quad \mathsf{R}(\nabla) \subseteq \mathsf{N}(0).$$

If instead of inclusions above, we have equalities, then we say that the complex (sequence) is *exact*. This is indeed the case for the simply connected domain $\Omega = (0, 1)$. By using the name *exact sequence*, we communicate more information: (a) the derivative of a function is zero *if and only if* the function is a constant, and (b) the function $\nabla : H^1(\Omega) \to L^2(\Omega)$ is a surjection (onto). From now on, we remove mention of the first and final terms of the exact sequence. The two spaces $\mathbb{R}$ and $\{0\}$, and the operators id and 0, are always assumed to buttress each of the sequences we later present. Moreover, whenever possible, we absorb the Ω assignment within the notation of each energy space. The domain Ω will always be assumed to be a simply connected domain in the relevant $\mathbb{R}^N$.

1D exact sequence. We now present the first exact sequence of simply connected domains in $\mathbb{R}$:

$$H^1 \xrightarrow{\nabla} L^2. \tag{1.1}$$

2D exact sequence. The exact sequence for simply connected domains in $\mathbb{R}^2$ is of the form

$$H^1 \xrightarrow{\nabla} H(\text{curl}) \xrightarrow{\nabla\times} L^2, \tag{1.2}$$

where $\nabla\times$ and $\times$ are understood in two dimensions:

$$\nabla \times E = \nabla \times \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \frac{\partial E_2}{\partial \xi_1} - \frac{\partial E_1}{\partial \xi_2}, \qquad E \times F = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \times \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = E_1F_2 - E_2F_1. \tag{1.3}$$