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# A discontinuous Galerkin method with Lagrange multiplier for hyperbolic conservation laws with boundary conditions

Mi-Young Kim

*Department of Mathematics, Inha University, Incheon 402-751, Republic of Korea*

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ABSTRACT

We introduce a discontinuous Galerkin method with Lagrange multiplier (DGLM) to approximate the solution to the hyperbolic conservation laws with boundary conditions. Lagrange multipliers are introduced on the edge/face of the element via weak divergence (Wang and Ye, 2014). The final global system has reduced numbers of unknowns of the standard DG methods. Numerical fluxes from finite volume/difference method are not considered. For the time discretization, backward Euler difference method is used. Stability of the approximate solution is proved in energy norm. Discontinuity of the solution is allowed in the error analysis. Local error estimates of $\mathcal{O}(h^{r+\frac{1}{2}}+\Delta t)$ with $P_r(E)$ elements ($r \geq \frac{d+1}{2}$) are derived, where $h$ and $\Delta t$ are the maximum diameter of the elements and time steps, respectively, and $d$ is the dimension of the spatial domain. The high order approximation is obtained under an appropriate condition on the stabilizing parameter. It is shown that the method preserves the property of the local mass conservation. An explanation on algorithmic aspects is given.



## 1. Introduction

We consider the following initial boundary value problem for the nonlinear hyperbolic conservation laws:

$$\partial_t u + \nabla \cdot \mathbf{F}(u) = 0 \quad \text{in } \Omega \times J, \tag{1.1}$$

$$u(\cdot, 0) = u^o \quad \text{in } \Omega, \tag{1.2}$$

$$u = g \quad \text{in } \partial\Omega \times J, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with $d = 1, 2$, or 3 and $J = (0, T]$ for $T > 0$. We assume that the data $g$ belongs to $L^2(\partial\Omega)$, $u^o$ belongs to $L^2(\Omega)$, and the flux $\mathbf{F}$ belongs to $(L^\infty(\Omega))^d$. The boundary condition in (1.1)–(1.3) has to be understood in a specific sense. For general flux $\mathbf{F}$ and in the context of entropy solutions, (1.1)–(1.3) has been first analyzed in the *BV* framework in [1]. The notion of entropy solution given there has been extended to the $L^\infty$ setting in [2,3].

In this paper, we consider (1.1)–(1.3) in the $L^2$ setting and develop a discontinuous Galerkin method with Lagrange multiplier (DGLM) to approximate the solution to the problem. We formulate a weak form by using locally defined weak divergence [4]. We introduce the Lagrange multipliers on each element. The approximate solution communicates only with the Lagrange multipliers at its edges/faces and it is solvable on each element in terms of the Lagrange multipliers. The discretized system is of block diagonal for the element unknowns and the reduced global system has block structure which is easily computable in parallel. Since the Lagrange multipliers are defined on the edge/faces only and single-valued on every edge/face, the final global system of the DGLM has fewer numbers of coupled unknowns than the usual DG methods.

*E-mail address:* mikim@inha.ac.kr.

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For the time discretization, we use backward Euler difference method. Stability of the approximate solution is then proved in the energy norm. It is shown that the method preserves local mass conservation on each element. Under a reasonable assumption on the regularity of the solution while discontinuity of the solution is allowed, we obtain the high order approximation with an appropriate choice of the stabilizing parameter. Numerical fluxes from the finite volume/difference method (FVM/FDM) are not considered. It is noted that the weak solutions of (1.1)–(1.3) produce a gap on the boundary between the data and the solution, which is usually given in implicit form [5]. Applying the DGLM with the stabilizing parameter sufficiently large, one can make the gap small enough.

Over the few decades, DG methods have been used widely in many fields with several appealing properties, even though the solvers can be expensive due to the number of unknowns. Examples of these schemes include the Bassi–Rebay method [6], the Local Discontinuous Galerkin (LDG) [7,8] methods, the Oden–Babuška–Baumann (OBB–DG) [9] method, and interior penalty Galerkin methods [10]. Many useful variants of the DG methods have been also developed to reduce the number of unknowns of the usual DG method. Examples include Coupling DG using mortar finite elements [11–13], Multiscale DG [14–17], and Hybridizable DG [18–20].

The DGLM method is a discontinuous Galerkin method having reduced numbers of unknowns. It applies the DG methodology to discretize the differential equation together with the introduction of the Lagrange multipliers on each element.

Recently, in [4], a weak Galerkin mixed finite element method was introduced together with the definition of the weak divergence to approximate the solution to the second order elliptic problems. There, the weak divergence was approximated via mixed finite elements.

On the other hand, the Lagrange multiplier has been introduced through the interface of the subdomains via domain decomposition method for coupling DG elements for the linear elliptic [11], for nonlinear parabolic [12], and for linear advection–diffusion–reaction problems [13]. There, the solutions were assumed to be smooth belonging to $H^{\frac{3}{2}+\epsilon}(\Omega)$ in the error analysis.

Hybridizable DG (HDG) methods have been developed for convection diffusion equations in [19,20]. Extensive studies to diverse problems such as Navier–Stokes and compressible/incompressible Euler equations have been also made by Cockburn et al. (See [18] and the references cited therein for details.) In those methods, approximate traces were introduced with appropriate choices of numerical fluxes from the FVM/FDM. However, to the best knowledge of the author, the error analysis was given only for the elliptic problems.

Multiscale DG methods have been developed in [14–17]. They have reduced numbers of unknowns of the DG methods. They use the solution space of the local problems as the approximate spaces.

In this paper, we develop a DGLM method by introducing a Lagrange multiplier on the edge/face of the element via weak divergence to approximate the solution to the nonlinear hyperbolic conservation laws. We here note that the idea of using Lagrange multipliers in conjunction with domain decomposition was first shown in the works of B. Fraeijs de Veubeke [21]. We use the approximate spaces consisting of discontinuous piecewise polynomials as usual DG methods. We allow the discontinuities of the solution in the error analysis and obtain the local error estimates of $\mathcal{O}(h^{r+\frac{1}{2}}+\Delta t)$ with $P_r(E)$ elements ($r \geq \frac{d+1}{2}$), where $h$ and $\Delta t$ are the maximum diameter of the elements and time steps, respectively, and $d$ is the dimension of the spatial domain. The high order approximation is obtained under an appropriate condition on the stabilizing parameter. To the best knowledge of the author, the error analysis of high order approximation for the nonlinear hyperbolic conservation laws allowing discontinuities of the solutions, this is the first result.

Concerning the stability and convergence of numerical schemes for conservation laws, most work were done for Cauchy problems. For smooth solutions of nonlinear conservation laws, under a restrictive time step $\Delta t < \gamma h^{\frac{4}{3}}$ for some constant $\gamma$ for high order element $r \geq 2$, a priori error estimates of $\mathcal{O}(h^{r+\frac{1}{2}} + (\Delta t)^2)$ for the second order explicit Runge–Kutta DG method, were obtained for general monotone numerical fluxes. Also, error estimates of $\mathcal{O}(h^{r+1} + (\Delta t)^2)$ were obtained for upwind numerical fluxes [22]. Recently, $L^2$-norm stability for the Cauchy problem of the scalar conservation laws and a priori error estimates for smooth solutions of scalar nonlinear conservation laws were obtained for the third order explicit Runge–Kutta DG methods. Quasi-optimal order for general monotone numerical fluxes and optimal order for upwind fluxes were obtained, for $r \geq 1$, under the standard CFL condition $\Delta t \leq \gamma h$ [23].

For nonsmooth solutions of nonlinear conservation laws, a cell entropy inequality for the semidiscrete DG method for the square entropy was proven [24], which implies that the numerical solutions, *if convergent*, will converge to an entropy solution to the Cauchy problem of the scalar conservation laws. An error estimate of $\mathcal{O}(h^{1/4})$ in the $L^1(\Omega)$-norm for the $P_0$ elements for a monotone finite volume scheme was proven [25]. For higher order $P_r$ elements, an error estimate with additional shock capturing terms to the method was also shown [25]. The result was improved to $\mathcal{O}(h^{1/2})$ for the explicit Lax–Friedrichs scheme [26]. There have been works on the a posteriori error estimates [27].

For initial boundary value problems for the nonlinear conservation laws, shock-capturing streamline-diffusion DG methods and the finite volume methods were considered in [28,29]. $L^\infty(L^\infty)$ boundedness and convergence of DG solutions were studied there. In [30,5], stable high order finite difference schemes and the entropy stable finite difference schemes were introduced. Problem of imposing stable boundary conditions on systems of conservation laws was also addressed.

The paper is organized as follows. In the next section we introduce some notations. In Section 3, we consider the steady state problem. We formulate the weak form for the DGLM. We establish equivalence between the weak formulation of the DGLM and the original problem (1.1)–(1.3). In Section 4, we introduce the DGLM and prove the stability of the approximate solution. We then show that the DGLM preserves the property of the local mass conservation. In Section 5, we study the

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