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Finite difference/finite element method for a nonlinear time-fractional fourth-order reaction–diffusion problem ☆

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ABSTRACT

In this article, a finite difference/finite element algorithm, which is based on a finite difference approximation in time direction and finite element method in spatial direction, is presented and discussed to cast about for the numerical solutions of a time-fractional fourth-order reaction–diffusion problem with a nonlinear reaction term. To avoid the use of higher-order elements, the original problem with spatial fourth-order derivative need to be changed into a second-order coupled system by introducing an intermediate variable $\sigma = \Delta u$. Then the fully discrete finite element scheme is formulated by using a finite difference approximation for time fractional and integer derivatives and finite element method in spatial direction. The unconditionally stable result in the norm, which just depends on initial value and source item, is derived. Some a priori estimates of L^2-norm with optimal order of convergence $O(\Delta_t^{2-\alpha}+h^{m+1})$, where Δ_t and h are time step length and space mesh parameter, respectively, are obtained. To confirm the theoretical analysis, some numerical results are provided by our method.

1. Introduction

The problems of fractional partial differential equations (FPDEs) have attracted a lot of attention. As we all know, based on the different position of fractional derivatives, FPDEs may be divided into three types including FPDEs with time, spatial and space–time fractional derivatives. In virtue of the difficulty for looking for the exact solutions of FPDEs, more and more scholars try to seek the numerical solutions by different numerical methods. These numerical methods mainly cover finite element (FE) methods [1–13], mixed finite element (MFE) methods [14,15], finite difference (FD) methods [16–38], finite volume (element) methods [25,39,40], (local) discontinuous Galerkin (L)DG methods [41–43], spectral methods [44–49] and so on.

Lin and Xu [46], Zhang and Xu [47], Lin et al. [48], and Zeng et al. [49] studied some spectral approximations for the time-fractional diffusion equation, time-fractional water wave model, the fractional Cable equation, Riesz space fractional reaction–diffusion equation, respectively. Baleanu et al. [16,44] presented some Laguerre spectral algorithms for fractional differential equations. Liu et al. [39], Zhuang [25], and Cheng [40] discussed some finite volume (element) methods for different fractional problems. Qiu et al. [41] proposed the nodal DG methods for 2D fractional diffusion equations, Deng and Hesthaven [42], Wei and He [43] solved some fractional differential equations by using LDG methods.

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So far, finite difference (FD) methods for solving fractional differential equations have been widely proposed and analyzed by many scholars. Yuste and Quintana-Murillo [17], Quintana-Murillo and Yuste [18,19], Chen et al. [24], Chen et al. [26], Sousa [28–30], Lin et al. [27], Huang et al. [37], Wang et al. [34], Wang and Wang [35], Li and Zeng [31], Li and Ding [36], Zhang et al. [33], and Gao and Sun [32] studied some FD schemes for the fractional (advection) diffusion or diffusion-wave equations. In [16], Baleanu et al. presented a central difference scheme for solving fractional optimal control problems. In [22], Meerschaert and Tadjeran solved two-sided space-FPDEs by using FD approximations. Atangana and Baleanu[20] proposed two FD schemes for fractional parabolic equations. Hu and Zhang [50] numerically solved fourth-order fractional diffusion-wave and subdiffusion systems by applying FD methods. Vong and Wang [38] discussed a high order compact FD scheme for time for fractional Fokker–Planck equations. Tadjeran et al. [21] proposed a second-order accurate FD scheme for the 1D fractional diffusion equations and Tadjeran and Meerschaert [23] considered the 2D case. Baeumera and Meerschaert [51] constructed a second-order Crank–Nicolson scheme, using a variant of the Grünwald FD formula for tempered fractional derivatives combined with a Richardson extrapolation. In [52] Chen and Deng proposed a second-order FD method for 2D two-sided space fractional convection diffusion equation. Recently, Sousa and Li [53] presented a weighted FD scheme for space fractional super-diffusion equation by introducing a second order approximation for the fractional Riemann–Liouville derivative of order α, $1 < \alpha \leq 2$. Based on the idea of weighted and shifted Grünwald difference (WSGD) operator, Tian et al. [54] proposed some second and third order approximations for Riemann–Liouville fractional derivatives for solving spatial fractional diffusion equations. Based on the idea of WSGD operator, Wang and Vong [55] proposed compact FD schemes for the modified time sub-diffusion equation with Riemann–Liouville fractional derivative and the temporal Caputo fractional diffusion-wave equation. Compared to schemes proposed previously, the temporal accuracy order of their schemes equals two. Following the idea of the WSGD operators [54,55] and using the equivalence of Caputo derivative and Riemann–Liouville derivative based on some regularity assumptions, Ji and Sun [56] presented a high-order compact difference scheme, in which the third-order accuracy formula was constructed for approximating Caputo time-fractional derivative. Gao et al. [57] achieved FD schemes with the global time second-order numerical accuracy, which does not depend on the values of α $(0 < \alpha < 1)$.

Finite element (FE) methods, which have been increasingly concerned by more and more people, are a kind of important numerical methods. In [3], Li et al. analyzed and discussed FE methods for nonlinear FPDEs with subdiffusion and superdiffusion. Zhao and Li [4] solved the time–space fractional telegraph equation by using FD/FE approximations. In [10], Li and Xu proposed a finite central difference/FE approximations with time second order convergence rate for 1D Caputo time-FPDE. Ford et al. [7] considered a FE method for time FPDEs, and proved the existence and uniqueness of the solutions by using the Lax–Milgram lemma. In [1], Zhang et al. studied the FE approximation combined with FD method for a 2D modified fractional diffusion equation. Jin et al. [12] considered semidiscrete Galerkin FE method and lumped mass Galerkin FE method for the homogeneous time-fractional diffusion equation, and obtained optimal error estimates with respect to the regularity of the solution. Jin et al. [13] studied two fully discrete FE schemes for fractional diffusion and diffusion-wave equations covering a Caputo fractional derivative in time, in which they established optimal error estimates with respect to the regularity of the initial data. In [58], Jin et al. considered a diffusion equation with multi-term time-fractional derivatives by using the Galerkin FE method. Bu et al. [8] solved 2D Riesz space fractional diffusion equations by using Galerkin FE method. Ma et al. [5] discussed the moving FE method for spatial FPDEs. Jiang and Ma [6] analyzed the results of a priori errors and calculated some numerical results based on some high-order elements for time-FPDEs. In [9], Li et al. developed FE methods for fractional Maxwell's equations. In [11], Zeng et al. used FE method combined with FD approximation for solving the time-fractional subdiffusion equation. In [14,15], Liu et al. proposed and analyzed two different MFE methods for two classes of linear FPDEs, respectively. From a great deal of literatures, we clearly see that FE methods have been used to look for the numerical solutions of FPDEs, especially they are applied to solving time FPDEs with second order spatial derivatives. But we can find from the current literatures that there are few studies on FE methods for solving time FPDEs with fourth-order spatial derivatives. Recently, Liu et al. [15] studied a linear time fractional fourth-order diffusion equation without reaction term by using FE method. The studied time fractional problem only covers a time fractional derivative, and the a priori estimates are arrived at based on mathematical induction. At the same time, we can find that the obtained time convergence rate in the theoretical analysis is only first-order $O(\Delta_t)$. However, we know that FE methods for solving the fractional fourth-order nonlinear problem with both time fractional and integer derivatives have been not found in the literatures.

In this article, our target is to present a FD/FE method to solve a nonlinear time-fractional fourth-order reaction–diffusion equation with first-order integer derivative in the time direction

$$\frac{\partial u}{\partial t} - \frac{\partial^\alpha \Delta u}{\partial t^\alpha} - \Delta u + \Delta^2 u = f(u) + g(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times J, \tag{1.1}$$

with boundary condition

$$u(\mathbf{x}, t) = \Delta u(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \partial\Omega \times \bar{J}, \tag{1.2}$$

and initial condition

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{1.3}$$

where Δ is the Laplacian operator, $\Omega \subset R^d (d \leq 2)$ and $J = (0, T]$ are a bounded convex polygonal domain with *Lipschitz* continuous boundary $\partial\Omega$ and the time interval with $0 < T < \infty$, respectively. $\frac{\partial^\alpha \Delta u}{\partial t^\alpha}$ is an anomalous sub-diffusion term,

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