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A new family of nonconforming finite elements on quadrilaterals[☆]

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ABSTRACT

In this paper, we generalize one first order nonconforming quadrilateral finite element proposed by Lin, Tobiska and Zhou to any odd order. We construct degrees of freedom for shape function spaces for this family of elements and show their unisolvency. In addition, we present a medium a priori error analysis for this family of nonconforming elements on general quadrilateral meshes. Compared with the classical error analysis of the nonconforming finite element method, the a priori analysis herein only needs the H^1 regularity of the exact solution. Numerics are presented to demonstrate theoretical results which in particular show dependence of convergence on mesh distortion parameters α.

1. Introduction

It is difficult to construct nonconforming quadrilateral finite elements of any order in a unifying way. Only very recently, Hu and Zhang generalized the nonconforming rotated Q_1 element due to Rannacher and Turek [1], the P_1 nonconforming element of [2,3] and the nonconforming cubic element of [4], and the nonconforming quadratic element of [5,6] to any order [7].

The enriched nonconforming rotated Q_1 element of [8] is an important element. Compared with the original nonconforming rotated Q_1 element, this enriched element has many new good properties. Besides superconvergence property [8], discrete eigenvalues produced by it for second order elliptic operators are smaller than exact ones when the meshsize is small enough [9,10]. One purpose of this paper is to generalize this element to any odd order, where shape function spaces are chosen as

$$E_m(\hat{K}) := \mathcal{P}_m(\hat{K}) + \mathrm{span}\{\xi^m\eta - \xi\eta^m, \xi^{m+1}, \eta^{m+1}\} \quad \text{for any } (\xi, \eta) \in \hat{K} := [-1, 1]^2,$$

for any odd integer $m > 0$. Here and throughout this paper, $\mathcal{P}_m(M)$ denotes the space of polynomials of degree $\leq m$ over the domain M. This new family can be regarded as a variant of that from [7].

The classic argument to bound the consistency error is to apply the Green theorem which requires that $u \in H^{\frac{3}{2}+\epsilon}(\Omega)$ with $\epsilon > 0$ so that the trace of the normal derivative of u along the edges of the triangulation is well defined. As the second purpose of this paper, we follow the idea of [11–14] to present a medium a priori error analysis for this family of nonconforming elements. However the method therein cannot be directly used herein since there is an extra average term, see Theorem 4.2 below. To analyze such an average term, we introduce a piecewise projection operator to decompose it into two parts, one vanishes due to the weak continuity of the functions in the nonconforming finite element spaces, the other can be bounded by the projection error. Compared with the classical error analysis of the nonconforming finite element method, the a priori analysis herein only needs the H^1 regularity of the exact solution u. In particular, this leads to the following error

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estimate

$$\|\nabla_h(u-u_h)\|_{0,\Omega} \lesssim h^{\min\left(r-1+\left[\frac{r+1}{2}\right](\alpha-1),r-1\right)}|u|_{H^s(\Omega)} + \left(\sum_{K\in T_h}\left(h_K^2\|f-\Pi_K f\|_{0,K}^2\right)\right)^{\frac{1}{2}},$$

provided that the exact solution $u \in H_0^1(\Omega)\cap H^s(\Omega)$ with $s > 1$, where $r = \min(m+1, s)$, u_h are the finite element solutions, f is the right-hand side, Π_K are some elementwise L^2 projection operators, $\alpha \geq 0$ are mesh distortion parameters. This estimate indicates that these elements will be optimal if $\alpha \geq 1$ and will not be optimal if $\alpha < 1$. In other words, these elements will lose accuracy on quadrilateral meshes with $\alpha < 1$. In particular, the first order element with $m = 1$ will not converge on quadrilateral meshes with $\alpha = 0$. We perform the first and third order elements on several families of meshes to study dependence of convergence on mesh distortion parameters α which confirm this estimate. Therefore, to get optimal approximations, we need to partition the domain under consideration into asymptotically parallelogram meshes with $\alpha \geq 1$. Fortunately, most of quadrilateral meshes used in practice satisfy this condition. By the theory from [15], an alternative way is to enrich shape function spaces on the reference element such that they contain $Q_m(\hat{K})$. In fact, we enrich the first order element to get a new element which converges optimally on all of these quadrilateral meshes under consideration in Section 5.4. Such an idea can be extended to high order elements. However, for brevity, we are not going to work on these generalizations in the current paper.

The rest of the paper is organized as follows. In the next section, we present degrees of freedom and show unisolvency. In Section 3, we define the nonconforming finite element spaces and the discrete problem for the Poisson equation. In Section 4, we analyze the a priori error estimate. In Section 5, we show numerical examples.

2. Degrees of freedom

For the space $E_m(\hat{K})(m = 2k+1)$, define degrees of freedom as follows

- moments of order $\leq 2k$ on each edge of $\hat{K}$;
- values at points of $\mathcal{I}_{2k-3}$;
- value of $\frac{\partial^{m+1}}{\partial\xi^{m+1}} + \frac{\partial^{m+1}}{\partial\eta^{m+1}}$.

Here and throughout this paper,

$$\mathcal{I}_{2k-3} := \{(\xi_\ell, \eta_\ell), \ell = 1, \ldots, (2k-1)(k-1)\}$$

be a set of interior points of $\hat{K}$ so that any polynomial $\hat{q}(\xi,\eta) \in \mathcal{P}_{2k-3}(\hat{K})$ can be uniquely defined by its values at points in $\mathcal{I}_{2k-3}$.

In the following, we let $\mathcal{L}_{2k+2}(\xi)$ and $\mathcal{L}_{2k+1}(\xi)$ denote Legendre polynomials with respect to $\xi \in [-1, 1]$ of degrees $2k+2$ and $2k+1$, respectively. Without loss of generality, we assume the coefficient of monomial ξ^{2k+1} in $\mathcal{L}_{2k+1}(\xi)$ and the coefficient of monomial ξ^{2k+2} in $\mathcal{L}_{2k+2}(\xi)$ are 1. In addition, we let the bubble function $\hat{b}(\xi,\eta) = (1-\xi^2)(1-\eta^2)$.

Before we show the unisolvency of these conditions, we prove the following preliminary result.

Lemma 2.1. *For any $\hat{v}(\xi,\eta) \in E_m(\hat{K})\backslash\mathrm{span}(\xi^{2k+2}+\eta^{2k+2})$, if*

$$\begin{aligned}
\hat{v}|_{\xi=-1} &= \alpha_1\mathcal{L}_{2k+2}(\eta) + \beta_1\mathcal{L}_{2k+1}(\eta),\\
\hat{v}|_{\eta=-1} &= \alpha_2\mathcal{L}_{2k+2}(\xi) + \beta_2\mathcal{L}_{2k+1}(\xi),\\
\hat{v}|_{\xi=1} &= \alpha_3\mathcal{L}_{2k+2}(\eta) + \beta_3\mathcal{L}_{2k+1}(\eta),\\
\hat{v}|_{\eta=1} &= \alpha_4\mathcal{L}_{2k+2}(\xi) + \beta_4\mathcal{L}_{2k+1}(\xi),
\end{aligned} \tag{2.1}$$

for some interpolation parameters $a_i, b_i,\ i = 1, \ldots, 4$, then

$$\hat{v}(\xi,\eta) = \hat{b}(\xi,\eta)\hat{q}(\xi,\eta) \quad \text{for some } \hat{q}(\xi,\eta) \in P_{2k-3}(\hat{K}).$$

Proof. We follow a similar idea of [7] to prove this result. We start with the expression of $\hat{v}(\xi,\eta)$ as

$$\hat{v}(\xi,\eta) = \hat{v}_1(\xi,\eta) + c_{2k+1,0}\xi^{2k+1} + c_{0,2k+1}\eta^{2k+1} + c_{2k+1,1}(\xi^{2k+1}\eta - \xi\eta^{2k+1}) + c_{2k+2,-}(\xi^{2k+2} - \eta^{2k+2}), \tag{2.2}$$

where $\hat{v}_1(\xi,\eta) \in \mathcal{P}_m(\hat{K})\backslash\mathrm{span}\{\xi^{2k+1}, \eta^{2k+1}\}$, and $c_{2k+1,0}, c_{0,2k+1}, c_{2k+1,1}$, and $c_{2k+2,-}$ are four interpolation parameters. Since the coefficient before monomial ξ^{2k+2} is opposite to that before monomial η^{2k+2} for $\hat{v}$, this gives

$$\alpha_1 = \alpha_3 = -\alpha_2 = -\alpha_4. \tag{2.3}$$

From (2.1), on the line $\xi = -1$, the coefficient for monomial η^{2k+1} is β_1. On the other hand, it follows from (2.2) that, on the line $\xi = -1$, the coefficient for monomial η^{2k+1} is $c_{0,2k+1} + c_{2k+1,1}$. It follows that

$$c_{0,2k+1} + c_{2k+1,1} = \beta_1.$$