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Asymptotically exact a posteriori error estimators for first-order div least-squares methods in local and global L_2 norm

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ABSTRACT

A new asymptotically exact a posteriori error estimator is developed for first-order div least-squares (LS) finite element methods. Let $(u_h, \boldsymbol{\sigma}_h)$ be LS approximate solution for $(u, \boldsymbol{\sigma} = -A\nabla u)$. Then, $\mathcal{E} = \|A^{-1/2}(\boldsymbol{\sigma}_h + A\nabla u_h)\|_0$ is asymptotically exact a posteriori error estimator for $\|A^{1/2}\nabla(u - u_h)\|_0$ or $\|A^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_0$ depending on the order of approximate spaces for $\boldsymbol{\sigma}$ and u. For $\mathcal{E}$ to be asymptotically exact for $\|A^{1/2}\nabla(u - u_h)\|_0$, we require higher order approximation property for $\boldsymbol{\sigma}$, and vice versa. When both $A\nabla u$ and $\boldsymbol{\sigma}$ are approximated in the same order of accuracy, the estimator becomes an equivalent error estimator for both errors. The underlying mesh is only required to be shape regular, i.e., it does not require quasi-uniform mesh nor any special structure for the underlying meshes. Confirming numerical results are provided and the performance of the estimator is explored for other choice of spaces for $(u_h, \boldsymbol{\sigma}_h)$.

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1. Introduction

The purpose of this paper is to introduce new, straightforward a posteriori error estimators for the least-squares (LS) finite element method for second order self-adjoint elliptic partial differential equations proposed in [1,2]. In these papers, the second-order equations are transformed into a system of first-order by introducing a new variable (flux) $\boldsymbol{\sigma} = -A\nabla u$. Least-squares methods based on the first-order system lead to a minimization problem, and the resulting algebraic equations involve a symmetric and positive definite matrix. One of the advantages of LS approaches is that it does not require inf–sup condition [3,4]. As a result, one can choose any conforming finite element spaces as approximate spaces. However, as was explained in [5], optimal rate of convergence for the flux in L_2-norm cannot be obtained without adding the redundant curl equation to the first-order system if the standard continuous piecewise polynomial spaces are used to approximate the dual variable $\boldsymbol{\sigma}$. On the other hand, with $H(\text{div})$ conforming spaces (such as the Raviart–Thomas (RT) spaces[6]) for the dual variable $\boldsymbol{\sigma}$, optimal rate of convergence is achieved for least-squares finite element methods [7]. Bochev and Gunzburger also noted the advantages of using RT spaces over the standard continuous piecewise polynomial spaces when a locally conservative approximation is essential [5,8]. With this as motivation, we will employ such approximation spaces in this paper.

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First-order LS methods approximate the primary variable u and dual variables $\boldsymbol{\sigma} = -A\nabla u$ simultaneously. In general, lowest order approximation spaces are used, i.e. piecewise linear polynomial spaces for u and RT_0 for $\boldsymbol{\sigma}$. However, this leads to approximation of the primary variable with $\mathcal{O}(h^2)$, while the dual variables $\boldsymbol{\sigma}$ are approximated with $\mathcal{O}(h)$. Hence, it is natural to consider different approximation spaces. Indeed, the error estimate in [7] indicates that using the lowest piecewise polynomial space for u and RT_1 for the dual variable approximate both variables with $\mathcal{O}(h^2)$. This motivates us to use different pair of approximations spaces and obtain a posteriori error estimates.

One of the advantages of div LS methods is that the LS functional can be used as an a posteriori error estimator for the natural energy norm. Recently, a modified version of the LS functional, where weight coefficients are introduced to scale the respective residuals, is proposed as a new a posteriori error estimator for these methods in the flux variable [9]. Our estimator uses only one term in the LS functional and the estimator turns out to be asymptotically exact with a certain choices of approximation spaces. Our estimator is of the following form:

$$\mathcal{E}(D) = \|A^{-1/2}(\boldsymbol{\sigma}_h + A\nabla u_h)\|_{0,D},$$

where $(u_h, \boldsymbol{\sigma}_h)$ is the LS solution for $(u, \boldsymbol{\sigma} = -A\nabla u)$ and $D \subseteq \Omega$ is the region of interest. Briefly, when $A\nabla u$ is approximated in higher order approximate spaces, then the estimator is asymptotically exact for $\|A^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,D}$ and when $\boldsymbol{\sigma}$ is approximated in higher order, then the estimate is asymptotically exact for $\|A^{1/2}\nabla(u - u_h)\|_{0,D}$. When both $A\nabla u$ and $\boldsymbol{\sigma}$ are approximated in the same order of approximate spaces, then the estimator is equivalent to the error under a mild assumption. Note that one of the advantages of LS methods is that they do not require the inf–sup condition. We use this advantage to choose appropriate approximation spaces for the primary function u and flux variable $\boldsymbol{\sigma}$. We will provide a detailed presentation in Section 4. In our numerical experiments in Section 5, we take $D = \Omega$, and $D = \tau$ where τ is a single element.

Recently, discontinuous Petrov Galerkin (DPG) method is proposed by Demkowicz and Gopalakrishnan [10,11]. Similar to LS approach, the method minimizes a residual of the governing equations in a certain norm. The DPG method has the possibility to locally compute a test space that is close to optimal. It would be interesting topic to modify the a posteriori error estimators developed in this paper for DPF method. We refer the interested readers to [12–18] and references therein for the DPG method and its applications to various problems.

The paper is organized as follows: Section 2 introduces mathematical equations for second-order scalar elliptic partial differential equations; the resulting div least-squares formulation for those equation is then described. In Section 3, we prescribe the finite element spaces and describe the basic properties of the corresponding least-squares approximate solutions. In Section 4, we propose a natural, asymptotically exact a posteriori error estimator for the flux variable $\boldsymbol{\sigma}$ and discuss the properties of the error estimator for different degree pairs of $(u_h, \boldsymbol{\sigma}_h)$. Also, we consider the case when the estimator is reliable and efficient under mild assumption. Finally, in Section 5 we provide numerical results that confirm the preceding analysis and discuss the usefulness of the estimator when asymptotic exactness does not hold.

2. Problem formulation

Let $H^s(\Omega)$ denote the Sobolev space of order s defined on Ω. For $s = 0$, $H^s(\Omega)$ coincides with $L_2(\Omega)$. We shall use the spaces

$$V = H_0^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\},$$
$$\mathbf{W} = H(\mathrm{div}) = \{\boldsymbol{\sigma} \in (L^2(\Omega))^n : \nabla\cdot\boldsymbol{\sigma} \in L^2(\Omega)\},$$

with norms $\|u\|_1^2 = (u, u) + (\nabla u, \nabla u)$ and $\|\boldsymbol{\sigma}\|_{H(\mathrm{div})}^2 = (\nabla\cdot\boldsymbol{\sigma}, \nabla\cdot\boldsymbol{\sigma}) + (\boldsymbol{\sigma}, \boldsymbol{\sigma})$.

Let Ω be a convex polygonal/polyhedral domain in $\mathbb{R}^n$, $n = 2, 3$, with boundary $\partial\Omega$. Consider

$$\begin{aligned} -\nabla\cdot A\nabla u + cu &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $A = (a_{ij})$ is uniformly symmetric positive definite, and a_{ij}, c and f are smooth functions. We assume the following *a priori* estimate:

$$\|u\|_{2+\delta} \le C\|f\|_{\delta}, \tag{2.2}$$

for some $\delta > 0$.

By introducing a new variable $\boldsymbol{\sigma} = -A\nabla u \in \mathbf{W}$, we transform the original problem into a system of first-order

$$\begin{aligned} &\boldsymbol{\sigma} + A\nabla u = 0 && \text{in } \Omega, \\ &\nabla\cdot\boldsymbol{\sigma} + c\,u = f && \text{in } \Omega, \\ &u = 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.3}$$

Then, the corresponding least-squares method for the system (2.3) is: Find $u \in V$, $\boldsymbol{\sigma} \in \mathbf{W}$ such that

$$\begin{aligned} b(u, \boldsymbol{\sigma}; v, \mathbf{q}) &\equiv (\nabla\cdot\boldsymbol{\sigma} + c\,u, \nabla\cdot\mathbf{q} + c\,v) + \left(A^{-1}(\boldsymbol{\sigma} + A\nabla u), \mathbf{q} + A\nabla v\right) \\ &= (f, \nabla\cdot\mathbf{q} + c\,v), \end{aligned} \tag{2.4}$$

for all $v \in V$, $\mathbf{q} \in \mathbf{W}$.

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