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A simple and efficient method with high order convergence for solving systems of nonlinear equations

Xiaoyong Xiao *, Hongwei Yin

Department of Mathematics, Nanchang University, 999 Xuefu Road, Nanchang, 330031, PR China

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ABSTRACT

We propose an $m+1$-step modified Newton method of convergence order $m+2$ to solve systems of nonlinear equations which are third Fréchet differentiable in a convex set containing the zero. Computational efficiency in the general form for a positive integer m is discussed, which shows that the efficiency increases with m when applied to large systems. Moreover, a comparison between the efficiency of this technique and some existing efficient methods is made, which implies that the present method is more efficient particularly for solving large systems of equations. Theoretical results about order of convergence and computational efficiency are largely verified in numerical examples.

1. Introduction

In numerical analysis and applied scientific branches, many researchers pay a great deal of attention to the construction of iterative methods for solving systems of nonlinear equations. For a given nonlinear system $F(x) : D \subseteq R^n \to R^n$, the question is how to find a vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^t$ such that $F(\alpha) = 0$, where

$$F(x) = (f_1(x), f_2(x), \ldots, f_n(x))^t, \quad \text{and} \quad x = (x_{(1)}, x_{(2)}, \ldots, x_{(n)})^t.$$

One simple choice for the approximation of the root α is the classical Newton's method of order two when the function F is continuously differentiable and a good initial approximation x_0 is given. The iterative method is defined by

$$x_{k+1} = x_k - F'(x_k)^{-1}F(x_k), \quad k = 0, 1, 2, \ldots, \tag{1.1}$$

where $F'(x)^{-1}$ is the inverse of first Fréchet derivative $F'(x)$ of the function $F(x)$.

So far, many modified Newton methods have been proposed to improve the order of convergence in the literature; for example, see [1–14] and references therein. Most of the existing methods are of order between two and six, then a natural question is whether we can construct an iterative method which achieves higher order convergence. Recently, it is worth mentioning that a number of novel methods are introduced for solving systems of nonlinear equations (see [15–24]).

On the other hand, the construction of higher order iterative methods should be based on low computational costs. Thus, the aim in developing iterative methods is to achieve as high as possible convergence order requiring as small as possible the evaluations of functions, derivatives and matrix inversions. We here propose a $m+1$-step method of convergence order $m+2$ using the predictor–corrector technique keeping the Jacobian $F'(x_k)$ unchanged in the whole process. The present technique

* Corresponding author.
E-mail address: xiaoxy31@gmail.com (X. Xiao).

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is simple but it has comparative advantage in both the speed of convergence and computational efficiency, particularly for large systems.

This paper is organized as follows. In Section 2, the $m+1$-step method is developed and detailed convergence analysis is given. In Section 3, computational efficiency is considered to compare with the other efficient methods. Several numerical examples are given in Section 4 to show the asymptotic behavior of the technique and to confirm the theoretical results. Finally, conclusions are made in Section 5.

2. The modified Newton method and its convergence

For a given nonlinear system $F(x)$, a well-known three-order iterative method [25,26] which improves the classical Newton's method is defined as

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k),\\ x_{k+1} &= y_k - F'(x_k)^{-1}F(y_k). \end{aligned} \tag{2.1}$$

Inspired by this, we want to find the other constants a and b such that the following construction is also of convergence order three:

$$\begin{aligned} y_k &= x_k - aF'(x_k)^{-1}F(x_k),\\ x_{k+1} &= y_k - bF'(x_k)^{-1}F(y_k). \end{aligned} \tag{2.2}$$

To work this out, we first recall the result of Taylor's expansion on vector functions (see [27]).

Lemma 1. *Let $F : D \subseteq R^n \to R^n$ be p time Fréchet differentiable in a convex set $D \subseteq R^n$, then for any $x, h \in R^n$, it holds that*

$$F(x+h) = F(x) + F'(x)h + \frac{1}{2!}F''(x)h^2 + \cdots + \frac{1}{(p-1)!}F^{(p-1)}(x)h^{p-1} + R_p, \tag{2.3}$$

where

$$\|R_p\| \le \frac{1}{p!}\sup_{0\le t\le 1}\|F^{(p)}(x+th)\|\,\|h\|^p \quad \text{and} \quad h^p = (h, h, \overset{p}{\cdots}, h).$$

Applying Lemma 1, we obtain the convergence property of the construction (2.2).

Lemma 2. *Let $F : D \subseteq R^n \to R^n$ be third Fréchet differentiable in a convex set $D \subseteq R^n$ containing the zero α of $F(x)$. Suppose that $F'(x)$ is nonsingular in α. Then, the sequence $\{x_k\}_{k\ge 0}$ $(x_0 \in D)$ obtained by construction (2.2) converges to α with order three provided that $a = \pm 1$ and $b = 1$.*

Proof. Since $F(\alpha) = 0$, then Taylor's expansion (2.3) for $F(x_k)$ about α is

$$F(x_k) = F'(\alpha)(x_k - \alpha) + \frac{1}{2!}F''(\alpha)(x_k - \alpha)^2 + O(\|x_k - \alpha\|^3).$$

Let $e_k = x_k - \alpha$, then we have

$$\begin{aligned} F(x_k) &= F'(\alpha)[e_k + A_2(e_k)^2 + O((e_k)^3)],\\ F'(x_k) &= F'(\alpha)[I + 2A_2e_k + O((e_k)^2)],\\ F'(x_k)^{-1} &= [I - 2A_2e_k + O((e_k)^2)]F'(\alpha)^{-1}, \end{aligned} \tag{2.4}$$

where $A_2 = \frac{1}{2!}F'(\alpha)^{-1}F''(\alpha)$ and $(e_k)^i = (e_k, e_k, \overset{i}{\cdots}, e_k)$, $e_k \in R^n$.

Let $\bar{e}_k = y_k - \alpha$. Applying (2.4), it is easy to see that

$$\bar{e}_k = e_k - aF'(x_k)^{-1}F(x_k) = (1-a)e_k + aA_2(e_k)^2 + O((e_k)^3), \tag{2.5}$$

which leads to

$$\begin{aligned} F'(x_k)\bar{e}_k &= F'(\alpha)[I + 2A_2e_k + O((e_k)^2)][(1-a)e_k + aA_2(e_k)^2 + O((e_k)^3)]\\ &= F'(\alpha)[(1-a)e_k + (2-a)A_2(e_k)^2 + O((e_k)^3)], \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} F(y_k) &= F'(\alpha)[\bar{e}_k + A_2(\bar{e}_k)^2 + O((\bar{e}_k)^3)]\\ &= F'(\alpha)[(1-a)e_k + (1-a+a^2)A_2(e_k)^2 + O((e_k)^3)]. \end{aligned} \tag{2.7}$$