Computers and Mathematics with Applications 69 (2015) 1242–1262

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Near-stability of a quasi-minimal surface indicated through a tested curvature algorithm

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ARTICLE INFO

Article history:
Received 26 March 2014
Received in revised form 5 March 2015
Accepted 8 March 2015
Available online 29 March 2015

Keywords:
Curvature algorithm
Bilinear interpolation
Variational improvement
Minimal surfaces

ABSTRACT

We decrease the *rms* mean curvature and area of a variable surface with a fixed boundary by iterating a few times through a curvature-based variational algorithm. For a boundary with a known minimal surface, starting with a deliberately chosen non-minimal surface, we achieve up to 65 percent of the total possible decrease in area. When we apply our algorithm to a bilinear interpolant bounded by four *non-coplanar* straight lines, the area decrease by the same algorithm is only 0.116179 percent of the original value. This relative stability suggests that the bilinear interpolant is already a quasi-minimal surface.

1. Introduction

In a variational problem, we write the form (termed *ansatz*) of a quantity to be minimized in such a way that what remains to be found is the value(s) of the variational parameter(s) introduced in the ansatz. An important application of the variational approach is to find characteristics of a surface (called a *minimal surface*) locally minimizing its area for a known boundary.

Minimal surfaces initially arose as surfaces of minimal surface area subject to some boundary conditions, a problem termed *Plateau's problem* [1,2] in variational calculus. Initial non-trivial examples of minimal surfaces, namely the catenoid and helicoid, were found by Meusnier in 1776. Later in 1849, Joseph Plateau showed that minimal surfaces can be produced by dipping a wire frame with certain closed boundaries into a liquid detergent. The problem of finding minimal surfaces attracted mathematicians like Schwarz [3] (who investigated triply periodic surfaces with emphasis on the surfaces called D (diamond), P (primitive), H (hexagonal), T (tetragonal) and CLP (crossed layers of parallels)), Riemann [1], Weierstrass [1] and R. Garnier [4]. Later, significant results were obtained by L. Tonelli [5], R. Courant [6,7], C.B. Morrey [8,9], E.M. McShane [10], M. Shiffman [11], M. Morse [12], T. Tompkins [12], Osserman [13], Gulliver [14], Karcher [15] and others.

The characteristic of a minimal surface resulting from a vanishing variation in its area is that its mean curvature should be zero *throughout*; see, for example, Section 3.5 of Ref. [16] for the partial differential equation (PDE) $H = 0$ this demand implies. This results in a prescription that to find a minimal surface we should find a surface for which the mean curvature function is zero at each point of the surface. Numerically, we have to find a zero for each set of values, which means we have to solve a large number of problems on a large grid. In converting this problem into a variational form we minimize the mean square *functional* of the mean curvature numerator i.e. μ_n^2 (given by Eq. (10)) with respect to our variational parameter. This

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http://dx.doi.org/10.1016/j.camwa.2015.03.015
0898-1221/

functional is more convenient than minimizing directly the area functional which has a square root in the integrand, but has the same extremal (i.e. a surface where this alternative functional is zero) as that of the area. This alternative can be compared with Douglas' suggestion of minimizing the Dirichlet integral that has the same extremal as the area functional. (A list of other possibilities of such functionals can be found in Refs. [17,18].) An additional advantage is that for μ_n^2 we know the value (that is, zero) to be achieved; for the area integral we do not know before calculations the target minimum value.

In an earlier work [19], for a boundary composed of four straight lines, we reduced the *rms* curvature and area of an initial surface chosen to be a bilinear interpolant. In this paper we took the ansatz for change in the surface to be proportional to the numerator of the mean curvature function for the surface; other factors in the change were the variational parameter, a function of the surface parameters (not to be confused with the variational parameters) whose form vanishes at the boundary and a vector assuring that we add a 3-vector to the original surface in 3-dimensions. In this paper we make an iterative use of the same ansatz, written in Eq. (6), to improve on this initial surface along with two other surfaces. Now we calculate afresh our variational parameter t for each iteration n. An important feature of our work is that at each iteration, the function we minimize (that is μ_n^2 of Eq. (10)), remains a polynomial in our variational parameter t. We used a computer algebra system to carry out our calculations and thus computational problems stopped us from actually implementing all the iterations. We somewhat avoided this impasse for a simpler case using a curve rather than a surface. This replaced the target flat surface by a straight line and the mean curvature by a second derivative. In this case too we had to eventually minimize a polynomial. In each case, the resulting value of the variational parameter specifies the curve or surface at each iteration.

In this paper, before reducing the area of the bilinear interpolant for which no minimal surface is completely known, we first considered two cases (a hemiellipsoid bounded by an elliptic curve and a hump-like surface spanned by four straight lines) where we already know a minimal surface for the boundary but we deliberately start with a non-minimal surface for the same boundary. We did this to find what fraction of the total possible area decrease (area of the starting non-minimal surface minus that of the known minimal surface) we achieve in the computationally manageable few iterations. If the target surface is flat, achieving it is sufficient but not necessary. This is because, at least according to the traditional definition of a minimal surface as a surface with zero mean curvature, for a fixed boundary we may have more than one minimal surface; vanishing mean curvature is a solution of an equation obtained by setting to zero the derivative of the surface with respect to the variational parameter in its modified expression and thus a solution surface can be guaranteed to be *only a locally minimal surface* in the set of all the surfaces generated by this modification and hence is not unique. Thus another question about our algorithm is whether or not it gets stuck in some other (locally) "minimal" curve or surface before it reaches the straight line or a flat surface. We tested our ansatz for two cases with known minimal surfaces; of course for neither case the initial surface we chose was a minimal one.

After testing in this way our algorithm, we used it to find a surface of smaller *rms* mean curvature and area for a boundary for which no minimal surface is known. For such a case we have tried to judge if an initial surface chosen with a fixed boundary is stable or quasi-stable against the otherwise decreasing areas that our algorithm can generate and took the resulting near stability as an indication that our initial surface is a quasi-minimal surface for our boundary. In this paper, we report area reductions in a number of iterations to strengthen our premise. We have compared our previous ansatz with a simpler alternative (see Eq. (25)) to point out that the ansatz we used can be repeatedly used, which is not the case with each possible ansatz, and thus is a non-trivial feature of the ansatz we used and are further using in this paper. The ansatz gives a significant reduction in areas in case of hemiellipsoid and hump-like surfaces (with already known minimal area) in computationally manageable few iterations whereas for the bilinear interpolant (a quasi-minimal surface of unknown minimal area) the decrease remains less than 0.1% of the original area. This also enabled us to numerically work out differential geometry related quantities for these surfaces. We have not been able to achieve a minimal surface within our computationally manageable resources. But, considering the possibility of implementing the same broad algorithm differently (maybe with a better computer program or with less reliance on the computationally demanding algebra system), our present work may be a test-bed for improved detailed algorithms aimed at computing minimal surfaces.

Our work can be compared to others [20,21] who have converted Dirichlet integrals into a system of linear equations which can be solved [20,18] to obtain extremals of a Dirichlet integral and thus surfaces of reduced area of a class of Bézier surfaces [22]. [18] employs the Dirichlet method and the extended blending energy method to obtain an approximate solution of the Plateau–Bézier problem by introducing a parameter λ in the Extended Dirichlet Functional (compare Eq. (4) of the Ref. [18] with our Eq. (10)). Determining this parameter λ gives all the inner control points obtained directly as the solution of a system of linear equation. In our case, determining the variational parameter t gives us variationally improved surfaces. The constraint that the mean curvature is zero is too strong and in most of the cases there is no known Bézier surface [18] with zero mean curvature.

Xu et al. [23] study approximate developable surfaces and approximate minimal surfaces (defined as the minimum of the norm of mean curvature) and obtain tensor product Bézier surfaces using a nonlinear optimization algorithm. Hao et al. [24] find the parametric surface of minimal area defined on a rectangular parameter domain among all the surfaces with prescribed borders using an approximation based on Multi Resolution Method using B-splines. Xu and Wang [25] study quintic parametric polynomial minimal surfaces and their properties. Pan and Xu [26] construct minimal subdivision surfaces with given boundaries using the mean curvature flow, a second order geometric PDE, which is solved by a finite element method. For some possible applications of minimal or quasi-minimal surfaces spanning bilinear interpolants and for a literature survey, one can read the introductory section of our work [19] on *Variational Minimization of String Rearrangement Surfaces* and [27] on *Coons Patch Spanning a Finite Number of Curves*.