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An optimal control problem for a two-prey and one-predator model with diffusion

N. Apreutesei [a], G. Dimitriu [b,*], R. Strugariu [a]

[a] Department of Mathematics and Informatics, "Gheorghe Asachi" Technical University, Bd. Carol I, nr. 11, Iaşi 700506, Romania

[b] Department of Mathematics and Informatics, "Grigore T. Popa" University of Medicine and Pharmacy, Str. Universităţii 16, Iaşi 700115, Romania

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ABSTRACT

An optimal control problem is studied for a reaction–diffusion system that models an ecosystem composed by one predator and two prey populations. One proposed to maximize the total density of the three populations. To do this, one proves the existence of an optimal solution and one establishes first and second order optimality conditions. Several numerical simulations performed in both one-dimensional and two-dimensional isolated environments and using different Holling type functional responses support the theoretical results.

1. Introduction

In this paper, we study an optimal control problem related to the reaction–diffusion system

$$\begin{cases} \dfrac{\partial y_1}{\partial t} = \alpha_1 \Delta y_1 + y_1 g_1(y_1) - y_1 y_2 f(y_1) \\ \dfrac{\partial y_2}{\partial t} = \alpha_2 \Delta y_2 - a y_2 + b y_1 y_2 f(y_1) + c y_2 y_3 h(y_3), \quad (t,x) \in Q = (0,T) \times \Omega \\ \dfrac{\partial y_3}{\partial t} = \alpha_3 \Delta y_3 + y_3 g_3(y_3) - y_2 y_3 h(y_3). \end{cases}$$

It models an ecosystem composed by a predator population of density y_2 and two prey populations, whose densities are y_1 and y_3. The three densities y_1, y_2, y_3 depend on the time $t \in [0, T]$ and on the spatial position $x \in \Omega$. The parameters $\alpha_1, \alpha_2, \alpha_3, a, b, c$ are positive and $\Omega \subseteq \mathbb{R}^d$ $(d \le 3)$ is a bounded domain with the boundary $\partial\Omega$ smooth enough. One supposes that the prey populations do not interact with each other, but any of them interacts with the predator.

Function $y_i g_i(y_i)$ is the intrinsic growth rate of the prey population i $(i = 1, 3)$ and signifies its growth rate in the absence of the predator. It can be linear if $g_i(y_i) = r_i$, logistic if $g_i(y_i) = r_i(1 - y_i/\kappa_i)$, Gompertz if $g_i(y_i) = g_{0i} \ln(\kappa_i/y_i)$ $(g_{0i}, r_i, \kappa_i > 0, i = 1, 3)$ etc. See [1,2] and the references therein.

The predator's functional responses to prey 1 and prey 3 are $y_1 f(y_1)$ and $y_3 h(y_3)$, respectively, and represent the number of prey individuals consumed per unit area and unit time per predator. They include as particular cases various classical functional responses: $y_1 f(y_1) = \beta y_1$ (Holling type I), $y_1 f(y_1) = \beta y_1/(1 + \mu y_1)$ (Holling type II), $y_1 f(y_1) = \beta y_1^2/(1 + \mu y_1^2)$

* Corresponding author.
E-mail addresses: napreut@gmail.com (N. Apreutesei), dimitriu.gabriel@gmail.com (G. Dimitriu), rstrugariu@gmail.com (R. Strugariu).

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(Holling type III; see [3]), $y_1 f(y_1) = \beta y_1/(\gamma + \mu y_1 + y_1^2)$ (Holling type IV; see [4]) etc. Here $\beta, \gamma > 0, \mu \geq 0$. Similar discussion can be presented for the functional response of the predator to prey 3. For the simplicity, we have assumed that f and h depend only on y_1 and on y_3 respectively, but the reasoning and the main results remain true also in the case when f and h depend on y_2 too. The term $-ay_2$ from the second equation shows the predator's mortality (parameter a is the per capita death rate of the predator).

In the above ecosystem we introduce a control u, which can be interpreted as a stimulant for the prey populations, i.e. one enhances their densities. For example, one can improve their food and life conditions. Assume that this enhancement is proportional to the density of each prey population. The proportionality factor u can be seen as a control variable and it is supposed to vary inside a finite interval [0, 1]. Hence the admissible set for the control u is

$$\mathcal{U} = \left\{u \in L^2(Q),\ 0 \leq u(t,x) \leq 1 \text{ a.e. on } Q\right\}.$$

Our goal is to maximize a certain weighted density of the three populations on Q and a weighted density on Ω at the end of the time interval $[0, T]$. More exactly, our problem can be formulated as

$$\min_u \Phi(y(u)) \quad \text{with } \Phi(y) = -\int_\Omega (l_1 y_1 + l_2 y_2 + l_3 y_3)(T,x)\,dx - \int_Q (k_1 y_1 + k_2 y_2 + k_3 y_3)(t,x)\,dt\,dx, \tag{1}$$

where $l_i, k_i > 0$ are given weights ($i = 1, 2, 3$), while $y = (y_1, y_2, y_3)$ is the solution of the controlled system

$$\begin{cases} \dfrac{\partial y_1}{\partial t} = \alpha_1 \Delta y_1 + y_1 g_1(y_1) + u\, y_1 - y_1 y_2 f(y_1) \\ \dfrac{\partial y_2}{\partial t} = \alpha_2 \Delta y_2 - a y_2 + b y_1 y_2 f(y_1) + c y_2 y_3 h(y_3), \\ \dfrac{\partial y_3}{\partial t} = \alpha_3 \Delta y_3 + y_3 g_3(y_3) + y_3 u - y_2 y_3 h(y_3) \end{cases} \tag{2}$$

for $(t, x) \in Q$, subject to some Neumann boundary conditions (i.e. the environment is isolated)

$$\frac{\partial y_i}{\partial \nu} = 0 \quad \text{on } \Sigma = [0,T] \times \partial\Omega,\ i = 1,2,3 \tag{3}$$

and to the initial conditions

$$y_i(0,x) = y_i^0(x), \quad x \in \Omega,\ i = 1,2,3. \tag{4}$$

The dependence $y(u)$ is defined by (2)–(4) with $u(\cdot)$ taken from the set $\mathcal{U}$.

We work under the following hypotheses on $f, h, g_1, g_3 : \mathbb{R} \to \mathbb{R}$ and y_i^0, $i = 1, 2, 3$.

(H1) g_1, g_3 are continuous and bounded on $(0, \infty)$;
(H2) f, h are continuous and positive on $(0, \infty)$ and bounded on bounded sets;
(H3) $y_i^0 \in H^2(\Omega)$, $y_i^0 > 0$ on Ω and $\partial y_i^0/\partial \nu = 0$ a.e. on $\partial\Omega$, for $i = 1, 2, 3$.

An optimal control problem for a predator–prey reaction–diffusion system with linear growth rate and linear predator's functional response was investigated in [5]. In [6] a control problem was analyzed for a nutrient–phytoplankton–zooplankton–fish system. Papers [7,8] are concerned with an ecosystem composed by a predator, a prey species and a plant. A diffusive two-competing-prey and one-predator model with Beddington–DeAngelis functional response was studied in [9]. Several population growth models depending implicitly or explicitly on resource dynamics have been mechanistically formulated in [2]. Taking into account a fixed amount of resources, the maximization of the net benefit in the conservation of a single species was studied in a control framework in [10]. The chaos control and synchronization problem of a three-species food chain model with Holling type I functional response was analyzed in [11]. Other models from population dynamics and optimal control problems can be found in [12,13,1,14–19].

The structure of the present paper is as follows. In Section 2, we use results from the semigroup theory and some well-known existence theorems from [20–22] to derive the global existence and uniqueness of a positive strong solution of the state equations (2), subject to the boundary conditions (3) and the initial conditions (4). We first associate a truncated problem, for which one can prove the existence. From the solution of the truncated problem, we derive a local solution of (2)–(4). We also show that this solution is bounded on its maximal set of definition, so it can be prolonged on the whole domain Q. As an alternative, we can prove another existence result using some general theorems from [23]. Section 3 is devoted to the existence of an optimal solution for problem (1)–(4). First order necessary optimality conditions are deduced in the next section. We write the dual system and the transversality conditions to show that the optimal control is bang–bang. Section 5 concerns with second order optimality conditions, and the Section 6 deals with numerical simulations which cover two particular cases of the functional responses considered before.

2. An existence theorem

We study the existence of the solution for problem (2)–(4), rewriting it like an abstract Cauchy problem in the Hilbert space $H = L^2(\Omega)^3$.