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Stability concepts and their applications

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ABSTRACT

The stability is one of the most basic requirement for the numerical model, which is mostly elaborated for the linear problems. In this paper we analyze the stability notions for the nonlinear problems. We show that, in case of consistency, both the N-stability and K-stability notions guarantee the convergence. Moreover, by using the N-stability we prove the convergence of the centralized Crank–Nicolson-method for the periodic initial-value transport equation. The K-stability is applied for the investigation of the forward Euler method and the θ-method for the Cauchy problem with Lipschitzian right side.

1. Introduction and motivation

In order to solve the operator equation, usually some numerical method is required, which means the construction of an adequate numerical model. One of the requirements for this model is stability, which seems to be one of the most challenging problems in numerical analysis. It is worth emphasizing that numerical stability is an intrinsic property of the numerical scheme and it is independent of the original continuous model. Commonly it is applied to the proof of convergence of the numerical method. (For this we need consistency, which establishes the link to the continuous problem.)

In the case of linear operators the first attempt was made by Kantorovich [1]. The theory for this case is worked out and it is widely known (e.g., [2,3]). However, the nonlinear theory is less elaborated. Stetter and Trenogin made the first attempts to define the notion of stability for nonlinear operators [4,5]. Later López-Marcos and Sanz-Serna began the systematic investigation of the basic numerical notions (consistency, stability and convergence) for nonlinear problems [6,7]. The abstract approach has stuck in. In the recent years we have made a similar approach to the investigation of the numerical solution of nonlinear operator equations in abstract settings. This work has been summarized in [8]. Thanks to these results and framework, we are able to use this approach to verify the stability of real-life problems. It is worth mentioning that there are other approaches to treat the stability notion for nonlinear problems (see for more details in [9]).

When we model some real-life phenomenon with a mathematical model, it results in a – usually nonlinear – problem of the form

$$F(u) = 0, \tag{1.1}$$

where $F : \mathcal{D} \to \mathcal{Y}$ is a (nonlinear) operator, $\mathcal{D} \subset \mathcal{X}$, $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces. In the theory of numerical analysis it is usually *assumed* that there exists a unique solution, which will be denoted by $\bar{u}$. Problem (1.1) can be given as a triplet $\mathscr{P} = (\mathcal{X}, \mathcal{Y}, F)$. We will refer to it as the *problem* $\mathscr{P}$.

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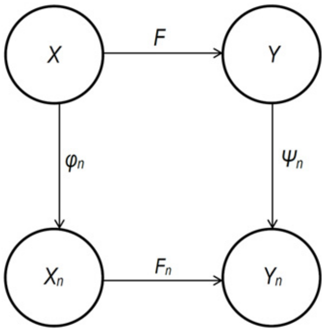

Fig. 1.1. The general scheme of numerical methods.

When we apply some numerical method, typically it generates a sequence of problems of the form

$$F_n(u_n) = 0, \quad n = 1, 2, \ldots, \tag{1.2}$$

where $\mathcal{X}_n$, $\mathcal{Y}_n$ are normed spaces, $\mathcal{D}_n \subset \mathcal{X}_n$ and $F_n : \mathcal{D}_n \to \mathcal{Y}_n$. If there exists a unique solution of (1.2), it will be denoted by $\bar{u}_n$. We define the mappings $(\varphi_n)_{n\in\mathbb{N}}$ from $\mathcal{X}$ into $\mathcal{X}_n$ and $(\psi_n)_{n\in\mathbb{N}}$ from $\mathcal{Y}$ into $\mathcal{Y}_n$, respectively.

Definition 1.1. The sequence of quintuples $\mathscr{D} = (\mathcal{X}_n, \mathcal{Y}_n, F_n, \varphi_n, \psi_n)_{n\in\mathbb{N}}$ is called a *discretization method*.

In sense of this definition we can illustrate the general scheme, showed in Fig. 1.1 (see, e.g. [10]).

For the convenience of the Reader, we formulate some basic definitions.

Definition 1.2. The element $e_n = \varphi_n(\bar{u}) - \bar{u}_n \in \mathcal{X}_n$ is called global discretization error. The element $l_n(v) = F_n(\varphi_n(v)) - \psi_n(F(v)) \in \mathcal{Y}_n$ is called local discretization error at the element v.

Clearly the local discretization error on the solution is $l_n(\bar{u}) = F_n(\varphi_n(\bar{u}))$.

Definition 1.3. We say that discretization $\mathscr{D}$ applied to the problem $\mathscr{P}$ is *convergent* if the relation

$$\lim \|e_n\|_{\mathcal{X}_n} = 0$$

holds.

Definition 1.4. The discretization $\mathscr{D}$ applied to problem $\mathscr{P}$ is called consistent on the element $v \in D$ if $\varphi_n(v) \in \mathcal{D}_n$ holds from some index and the relation

$$\lim \|l_n(v)\|_{\mathcal{Y}_n} = 0$$

holds.

In numerical analysis one of the most important tasks is to guarantee the convergence of the sequence of the numerical solutions to the true solution $\bar{u}$. Generally, consistency in itself is not enough, therefore, to guarantee the convergence, we need certain additional condition. This is the notion of stability.

First of all we consider the sequence of linear problems, i.e., the problems

$$L_n(u_n) = 0, \quad n = 1, 2, \ldots, \tag{1.3}$$

where for each n the operator L_n is linear and $L_n : \mathcal{D}_n \to \mathcal{Y}_n$. Naturally, we always assume the solvability of the problems (1.3), i.e., the existence of the operators $L_n^{-1} : \mathcal{Y}_n \to \mathcal{D}_n$. In this case, as it is known, the linear stability requires that $\left\|L_n^{-1}\right\|_{Lin(\mathcal{Y}_n, \mathcal{X}_n)} \le S$ holds, where S is some positive constant.

Then the consistency and the stability together ensure the convergence. This result is well-known as the Lax (or sometimes Lax–Richtmyer–Kantorovich [2]) theorem. In numerical analysis it is also called as the "basic theory of numerical analysis".

2. Generalization of the stability notion

The linear stability notion implies some basic results. However, obtaining these consequences, we exploit the linearity of the operators L_n. In the rest of the paper our main aim is to study how to define the notion of stability in a suitable way for general (nonlinear) case.

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