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Splitting schemes for poroelasticity and thermoelasticity problems[☆]

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ABSTRACT

In this work, we consider the coupled systems of linear unsteady partial differential equations, which arise in the modelling of poroelasticity processes. Stability estimates of weighted difference schemes for the coupled system of equations are presented. Approximation in space is based on the finite element method. We construct splitting schemes and give some numerical comparisons for typical poroelasticity problems. The results of numerical simulation of a 3D problem are presented. Special attention is given to using high performance computing systems.

1. Introduction

Poroelasticity [1–4] and thermoelasticity problems [5–8] are mathematically and physically analogous due to the fact that the pressure and temperature play a similar role in deformation of a body. For instance, a change in the temperature or a change in the pressure in a body results in equal normal strains in three orthogonal directions and no shear strains.

The basic mathematical models of such problems include the Lame equation for the motion and the pressure/temperature equations. The fundamental point is that the system of equations is coupled: the equation for the motion includes the volume force, which is proportional to the temperature/pressure gradient, and the temperature/pressure equations include a term, which describes the compressibility of the medium.

To solve numerically the coupled quasi-stationary linear system of equations, we approximate our system using the finite element method [9,10]. Variational formulations for poroelasticity and thermoelasticity problems and finite element approximations are considered in [11–13]. For the considered system of equations the finite element interpolation with equal order spaces for both the displacement and pressure/temperature fields is not inf–sup stable. It is well known that in classical mixed formulations, the finite element spaces must satisfy the LBB stability conditions [14–16]. This type of discretizations for poroelasticity problems provides a lower order of convergence for the pressure in comparison with the displacements.

Quasi-stationary problem (steady for motion and unsteady for the temperature/pressure) are solved using the weighted scheme [17,18]. The stability analysis is performed [19,20] in the framework of the general theory of stability for operator-difference schemes [21,22].

At present, different classes of additive operator-difference schemes for evolutionary equations are constructed via an additive representation of the main operator. Additive schemes are constructed using splitting; they are associated with

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transition to a new time level on the basis of the solution of simpler problems for individual operators in the additive decomposition [23,24]. We consider splitting (additive) schemes for poroelasticity/thermoelasticity problems with additive representation of the operator at the time derivative; we also consider modifications of splitting schemes. Similar schemes are examined in many studies [25–28], but in our work, we show that these schemes are no more than regularization schemes [24,29]. From much work on numerical methods for poroelasticity problems, we also note the work [30], where the problems of approximation in time are discussed. The questions of construction of efficient solvers for the solution of the discrete problem on each new layer are considered in [30,31].

The work is organized as follows. Section 2 provides the mathematical model for the poroelasticity problem, which is the same as for the thermoelasticity problem. In Section 3, we consider the properties of the differential problem and give a priori estimates for the stability for the solutions with respect to the initial data and the right-hand side. In this section, we also formulate our problem as an initial value problem for a system of linear ordinary differential equations. Discretizations in space and time are performed in Section 4. Here we present some analysis of the weighted differential schemes for coupled system. The central part of the work deals with the construction and numerical comparison of the stability of the splitting schemes. In Section 5, we consider the additive (splitting) schemes that are compared for typical poroelasticity problems in Section 7. Some modifications for additive (splitting) schemes associated with regularization are introduced in Section 6. The results of numerical simulation of a 3D problem on high performance computing systems are presented in Section 8.

2. Problem formulation

The linear poroelasticity equations can be expressed [1] as

$$\begin{aligned} &\operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}) - \alpha \operatorname{grad} p = 0, \\ &\alpha \frac{\partial \operatorname{div} \boldsymbol{u}}{\partial t} + \beta \frac{\partial p}{\partial t} - \operatorname{div} \left(\frac{k}{\nu} \operatorname{grad} p \right) = f(\boldsymbol{x}, t), \end{aligned} \tag{1}$$

with the boundary conditions

$$\begin{aligned} &\boldsymbol{\sigma n} = 0, \quad \boldsymbol{x} \in \Gamma_N^u, \qquad \boldsymbol{u} = 0, \quad \boldsymbol{x} \in \Gamma_D^u, \\ &-\frac{k}{\nu}\frac{\partial p}{\partial n} = 0, \quad \boldsymbol{x} \in \Gamma_N^p, \qquad p = g, \quad \boldsymbol{x} \in \Gamma_D^p, \end{aligned} \tag{2}$$

and the initial conditions

$$p(\boldsymbol{x}, 0) = p_0, \quad \boldsymbol{x} \in \Omega. \tag{3}$$

Here the primary variables are the fluid pressure p and the displacement vector $\boldsymbol{u}$. Also, $\boldsymbol{\sigma}$ is the stress tensor, $\beta = 1/M$, M is the Biot modulus, k is the permeability, ν is the fluid viscosity, α is the Biot–Willis fluid/solid coupling coefficient and f is a source term representing injection or production processes. Body forces are neglected, $\boldsymbol{n}$ is the unit normal to the boundary.

The stress tensor is given by

$$\boldsymbol{\sigma} = 2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda \operatorname{div} \boldsymbol{u} \boldsymbol{I}, \tag{4}$$

where $\boldsymbol{I}$ is the identity tensor and $\boldsymbol{\varepsilon}$ is the strain tensor:

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}\left(\operatorname{grad} \boldsymbol{u} + \operatorname{grad} \boldsymbol{u}^T\right),$$

and μ, λ are Lame parameters:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)},$$

where E is the Young's modulus and ν is the Poisson's ratio.

In the case of thermoelasticity, the governing equations are the same as Eqs. (1) [5–8] with the temperature T instead of the pressure:

$$\begin{aligned} &\operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}) - \beta \operatorname{grad} T = 0, \\ &\beta T_0 \frac{\partial \operatorname{div} \boldsymbol{u}}{\partial t} + c \frac{\partial T}{\partial t} - \operatorname{div} (\kappa \operatorname{grad} T) = 0, \end{aligned} \tag{5}$$

where c is the heat capacity of the unit volume in the absence of deformation, κ is the thermal conductivity and β is the coupling coefficient playing the similar role as the Biot–Willis coefficient α. Here T_0 is the constant initial temperature of a medium.

3. Properties of the differential problem

We define the standard Hilbert space $H = L_2(\Omega)$ for the pressure with the following inner product and norm:

$$(u, v) = \int_\Omega u(\boldsymbol{x})\, v(\boldsymbol{x})\, dx, \quad \|u\| = (u, u)^{1/2}$$

and the Hilbert space $\boldsymbol{H} = (L_2(\Omega))^d$ for the displacements. Here $d = 2, 3$ is the number of spatial dimensions.