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The minimum cost perfect matching problem with conflict pair constraints

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ABSTRACT

In this paper we address the minimum cost perfect matching problem with conflict pair constraints (MCPMPC). Given an undirected graph G with a cost associated with each edge and a conflict set of pairs of edges, the MCPMPC is to find a perfect matching with the lowest total cost such that no more than one edge is selected from each pair in the conflict set. MCPMPC is known to be strongly $\mathcal{NP}$-hard. We present additional complexity results and identify new polynomially solvable cases for the general MCPMPC. Several heuristic algorithms and lower bounding schemes are presented. The proposed algorithms are tested on randomly generated instances. Encouraging experimental results are also reported.

1. Introduction

A matching M in an undirected graph G is defined as a set of edges such that no two edges of M are incident on the same node. The matching problem and its several variations are well studied in the combinatorial optimization literature. Some variations of the matching problem such as the *maximum cardinality matching problem* (*MCMP*) and the *minimum cost perfect matching problem* (*MCPMP*) are polynomially solvable while variations such as the *quadratic assignment problem* (*QAP*) and the *assignment problem with budget constraints* (*APBC*) are $\mathcal{NP}$-hard. For a thorough review on the bipartite matching problem and its variations, we refer to the book by Burkard et al. [1].

Another variation of the matching problem is the *minimum cost perfect matching problem with conflict pair constraints* (*MCPMPC*) introduced by Darmann et al. [2]. To the best of our knowledge, there are no other published papers discussing the MCPMPC. In addition to an undirected graph G with edge costs, the definition of MCPMPC uses a *conflict set*, which consists of edge pairs that are incompatible in a feasible solution. These edge pairs are called *conflict pairs* and they can alternatively be represented by a *conflict graph* $\hat{G}$, in which the nodes correspond to edges of G that are part of the conflict pairs and each edge of $\hat{G}$ represents a conflict pair [3]. Since the conflict pairs result in binary disjunctive constraints, the MCPMPC can be used in the applications of matching problems where incompatibilities exist between some pairs of edges. Also the MCPMPC arises as a subproblem when solving the *quadratic bottleneck assignment problem* (*QBAP*) [4], which is a generalization of the *bandwidth minimization problem* in matrices and graphs [5].

It is proved by Darmann et al. [2] that the MCPMPC is strongly $\mathcal{NP}$-hard on a general graph G even if the conflict graph is a collection of single edges. In this paper we continue exploring complexity results for the MCPMPC. We show that when the conflict graph is arbitrary but the original graph G is a collection of disjoint 4-cycles, then the problem is also $\mathcal{NP}$-hard. Various polynomially solvable cases are then identified by restricting the structures of G and $\hat{G}$.

Since the MCPMPC can be formulated as an MCPMP with additional constraints, any algorithm available to solve such problems can be used directly to solve the MCPMPC, as well. Some such references include [6,7] consider the special case when G is bipartite. However, these algorithms are not good at handling a large number of additional constraints and therefore in this paper, we investigate other approaches to solve the MCPMPC. We have five heuristic algorithms including a frequency guided tabu search algorithm and a genetic mutation guided tabu search algorithm, both make use of a general purpose mixed integer linear programming (MILP) solver in each enhanced search iterations. These two heuristics are shown to be computationally promising and similar ideas can easily be incorporated in tabu search based heuristics for other combinatorial optimization problems. Lower bounding approaches for the MCPMPC are also considered by using formulation transformation, Lagrangian relaxation (LR), and linear programming (LP) relaxation, respectively. Finally, the proposed heuristics and lower bounding schemes are tested on randomly generated instances and report the experimental results.

The rest of the paper is organized as follows. In Section 2 we give two integer programming formulations of MCPMPC, prove

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new complexity results, and present new polynomially solvable special cases. Heuristics and lower bounding schemes for the MCPMPC are presented in Section 3, followed by the computational results in Section 4. Finally, concluding remarks are given in Section 5.

2. Complexity and polynomially solvable cases

Let $G=(V,E)$ be an undirected graph such that $|V|=n$ and $|E|=m$. For each edge $e\in E$, a cost c_e is defined. A *conflict set* $P\subseteq\{\{e,f\}: e,f\in E, e\neq f\}$ is also given and elements of P are called *conflict pairs*. Let $\mathcal{M}$ be the family of all perfect matchings in G. The *incidence vector* $x=(x_1,x_2,\dots,x_m)$ of a perfect matching M is defined as

$$x_e=\begin{cases}1 & \text{if } e\in M,\\ 0 & \text{otherwise.}\end{cases}$$

Thus a matching is completely represented by its incidence vector. Let $F(M)$ be the convex hull of incidence vectors of $M\in\mathcal{M}$. Then the MCPMPC can be formulated as the following *integer linear programming* (ILP) problem:

$$\min \quad z=\sum_{e\in E}c_e x_e$$

s.t.

$$x\in F(\mathcal{M}),$$

$$x_e+x_f\le 1 \quad \text{for } \{e,f\}\in P, \tag{1}$$

$$x_e\in\{0,1\} \quad \text{for } e\in E. \tag{2}$$

Constraints (1) enforce that no edge pair in a feasible perfect matching can be a conflict pair. In other words, if $\{e,f\}$ is a conflict pair then at most one of edges e or f can appear in a matching. The conflict set P can also be represented conveniently as a *conflict graph* $\hat{G}=(\hat{V},\hat{E})$, where $\hat{V}=E$ and $(e,f)\in\hat{E}$ if and only if $\{e,f\}\in P$.

In fact, it is not necessary to choose $\hat{V}=E$. Let

$$E^*=\{e: e\in E \text{ and } e \text{ is in some conflict pair of } P\}.$$

Then we can choose the node set of $\hat{G}$ as E^* since if $\hat{V}=E$, then nodes in $\hat{G}$ corresponding to edges in $E-E^*$ are isolated nodes in $\hat{G}$ and dropping them does not affect the problem definition. Further, any super set of E^* can also be chosen as $\hat{V}$. This observation is convenient in defining conflict graphs in some situations.

The MCPMPC can also be formulated as a *quadratic minimum cost perfect matching problem* (*QMCPMP*) where a cost a_{ef} is used for the edge-pair (e,f). Define

$$a_{ef}=\begin{cases}c_e & \text{if } e=f,\\ \Psi & \text{if } \{e,f\}\in P,\\ 0 & \text{otherwise,}\end{cases}$$

where Ψ is a large number. Then the QMCPMP formulation of MCPMPC is given by

$$\min z=\sum_{e\in E}\sum_{\substack{f\in E\\ e\neq f}}a_{ef}x_e x_f+\sum_{e\in E}a_{ee}x_e$$

s.t.

$$x\in F(\mathcal{M}),$$

$$x_e\in\{0,1\} \quad \text{for } e\in E.$$

As mentioned in Section 1, Darmann et al. [2] showed that the MCPMPC is strongly $\mathcal{NP}$-hard even if the conflict graph is a collection of single edges. This result rules out (unless $\mathcal{P}=\mathcal{NP}$) the possibility of getting polynomially solvable special cases of the MCPMPC by restricting the structure of the conflict graph to any reasonable non-trivial sparse graph. We thus consider a related question where the topology of G is restricted, instead of the structure of the conflict graph.

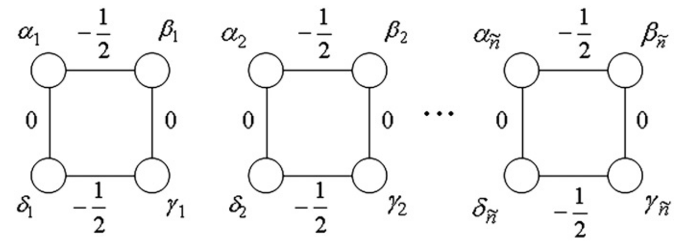

Fig. 1. G' as a collection of disjoint 4-cycles constructed from $\tilde{G}$.

Theorem 1. *The MCPMPC is $\mathcal{NP}$-hard even if G is a collection of disjoint 4-cycles.*

Proof. We reduce the maximum independent set problem to an MCPMPC on a collection of 4-cycles. Let $\tilde{G}=(\tilde{V},\tilde{E})$ be a given graph with $\tilde{V}=\{1,2,\dots,\tilde{n}\}$. For each node $i\in\tilde{V}$, create a 4-cycle $\alpha_i-\beta_i-\gamma_i-\delta_i-\alpha_i$. Let the cost of (α_i,β_i) and (γ_i,δ_i) be $\frac{-1}{2}$ and the cost of (β_i,γ_i) and (δ_i,α_i) be 0. The resulting graph G' is shown in Fig. 1. Note that any perfect matching in G' must select either both edges (α_i,β_i) and (γ_i,δ_i), or both edges (α_i,δ_i) and (β_i,γ_i).

Let

$$P=\{\{(\alpha_i,\beta_i),(\gamma_j,\delta_j)\},\{(\alpha_i,\beta_i),(\alpha_j,\beta_j)\},\{(\gamma_i,\delta_i),(\alpha_j,\beta_j)\},\{(\gamma_i,\delta_i)(\gamma_j,\delta_j)\}:(i,j)\in\tilde{E}\}$$

be the conflict set. Thus we have constructed an instance of the MCPMPC. It is not difficult to verify that the maximum independent set in $\tilde{G}$ is of size k if and only if the MCPMPC has optimal objective function value $-k$. Since the maximum independent set problem is $\mathcal{NP}$-hard [8], the MCPMPC on a collection of disjoint 4-cycles is also $\mathcal{NP}$-hard. □

Corollary 1. (1) *The maximization version of the MCPMPC on a collection of disjoint 4-cycles is $\mathcal{NP}$-hard.*

(2) *The MCPMPC* (*minimization/maximization*) *cannot be approximated within a factor of $m^{1-\epsilon}$, for any $\epsilon>0$, even if G is a collection of disjoint 4-cycles.*

Proof. By replacing the edge costs $\frac{-1}{2}$ by $\frac{1}{2}$ and following similar arguments as in the proof of Theorem 1, (1) follows. The proof of (2) is also straightforward since the reduction discussed in the proof of Theorem 1 and Corollary 1(1) is approximation ratio preserving and it is well known that the maximum independent set problem on a graph with m edges cannot be approximated within a factor of $m^{1-\epsilon}$ for any $\epsilon>0$ [8]. □

Related complexity results on 2-ladder conflict graphs are discussed in [2].

Next we consider a graph $G=(V,E)$ with the following properties:

(P1) There exists subgraphs $G_1,G_2,\dots,G_k$ such that any perfect matching M in G is a union of perfect matchings M_i in G_i, i.e. $M=\bigcup_{i=1}^{k}M_i$.

(P2) Each G_i has a specified edge e_i such that e_i does not conflict with any of the edges of G_i and the conflict set $P\subseteq\{e_1,e_2,\dots,e_k\}\times\{e_1,e_2,\dots,e_k\}$, i.e. there are no conflict pairs containing edges in $E\backslash\{e_1,e_2,\dots,e_k\}$. Thus the vertex set of $\hat{G}$ can be viewed as $\{e_1,e_2,\dots,e_k\}$.

Fig. 2 gives an example of a graph G and the associated conflict graph $\hat{G}$ satisfying properties (P1) and (P2).