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A Geometric Programming Approach for Bivariate Cubic L_1 Splines

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Abstract—Bivariate cubic L_1 splines provide C^1-smooth, shape-preserving interpolation of arbitrary data, including data with abrupt changes in spacing and magnitude. The minimization principle for bivariate cubic L_1 splines results in a nondifferentiable convex optimization problem. This problem is reformulated as a generalized geometric programming problem. A geometric dual with a linear objective function and convex cubic constraints is derived. A linear system for dual-to-primal conversion is established. The results of computational experiments are presented.

Keywords—Cubic L_1 spline, Geometric programming, Interpolation, Spline function, Bivariate.

1. INTRODUCTION

Conventional polynomial splines have been highly successful for interpolation and approximation of smoothly spaced data lying on smooth surfaces [1–5]. In classical metrics, they have excellent approximation power, their coefficients can be calculated by efficient, band-matrix-based algorithms, and their locally polynomial nature ensures efficient evaluation. However, for multiscale data, that is, data with abrupt changes in spacing and/or magnitude, conventional polynomial splines do not "preserve shape well," a requirement that was not included in the classical concept

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of good approximation power [6–8]. Since the vast majority of objects that occur in modern geometric representation and visualization systems—natural and urban terrain, animals, plants, other biological objects, many mechanical objects, etc.—are irregular objects represented digitally by multiscale data, conventional polynomial splines are not widely used in these systems. Although alternatives such as rational splines (NURBS) and wavelets have gained increasing acceptance, most of these objects are still represented by piecewise flat surfaces on irregular triangulations.

Shape preservation has not yet been defined quantitatively, but it is generally agreed that it involves avoiding extraneous "nonphysical" oscillation and artifacts. For example, one normally prefers that a lake with a flat surface and a piecewise planar roof of a building be represented without permanent, nonphysical oscillation. Recently, a new class of polynomial splines, cubic L_1 splines, that do preserve shape well has been developed [6–8]. In 2002, Cheng *et al.* [9] proposed a geometric programming framework [10–12] for univariate cubic L_1 splines. The present paper extends the geometric programming framework to bivariate cubic L_1 interpolating splines. This extension is not only practically important, but also technically nontrivial. In real-life applications, the univariate case is very rare and serious scenarios begin with the bivariate case. The bivariate theory must take into account the structure of a two-dimensional grid, whereas the univariate theory is formulated using a less complex one-dimensional grid.

In the remainder of this paper, bivariate cubic L_1 interpolating splines will be called, for short, L_1 splines. The coefficients of L_1 splines are calculated by minimizing the L_1 norm of second partial derivatives of candidate C^1-smooth piecewise cubic surfaces. Calculating the coefficients of an L_1 spline is equivalent to solving a nondifferentiable convex optimization problem. In Section 2, a precise definition of and information about L_1 splines are given. In Section 3, a geometric programming framework for L_1 splines is set up. This framework includes three parts, namely, the primal problem, the dual problem, and the dual-to-primal transformation. In Section 4, we present computational results. Section 5 provides a summary, discussion of algorithmic issues, and information about future research directions.

2. L_1 SPLINES BASED ON SIBSON ELEMENTS

In the present paper, we consider L_1 splines on a tensor-product grid. Let there be given a *tensor-product grid set* $\Delta = \{\{x_i\}_{i=0}^{I}, \{y_j\}_{j=0}^{J}\}$ with monotonic but not necessarily equally spaced knots

$$\begin{aligned} a = x_0 < x_1 < \cdots < x_{I-1} < x_I = b, \\ \tilde{a} = y_0 < y_1 < \cdots < y_{J-1} < y_J = \tilde{b}. \end{aligned} \tag{1}$$

The set Δ partitions the domain $D \triangleq [a,b]\times[\tilde{a},\tilde{b}]$ into $I\times J$ rectangles, $T_{ij} = [x_i, x_{i+1}]\times[y_j, y_{j+1}]$, $i = 0,\dots,I-1$, $j = 0,\dots,J-1$. For the x and y dimensions of T_{ij}, we adopt the notation

$$h_i^x = x_{i+1} - x_i, \qquad h_j^y = y_{j+1} - y_j. \tag{2}$$

The data to be interpolated will be (x_i, y_j, z_{ij}), $i = 0,\dots,I$, $j = 0,\dots,J$.

In this paper, we will analyze L_1 splines constructed using Sibson elements. In this section, we first describe Sibson elements and then define and characterize L_1 splines.

2.1. Sibson Elements

To generate Sibson elements [13], one begins by dividing each rectangle T_{ij} into four triangles by drawing the two diagonals of the rectangle. The triangles are labeled 1, 2, 3, and 4 and are denoted by T_{ij}^k, $k = 1,2,3,4$, respectively, as shown in Figure 1.

DEFINITION 2.1. SIBSON ELEMENT. *A Sibson element $z(x,y)$ on a rectangle T_{ij} is a function that*

(i) *is a cubic polynomial on each of the four triangles T_{ij}^k, $k = 1,2,3,4$, in the rectangle,*