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Branch and bound for the cutwidth minimization problem

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ABSTRACT

The cutwidth minimization problem consists of finding a linear arrangement of the vertices of a graph where the maximum number of cuts between the edges of the graph and a line separating consecutive vertices is minimized. We first review previous approaches for special classes of graphs, followed by lower bounds and then a linear integer formulation for the general problem. We then propose a branch-and-bound algorithm based on different lower bounds on the cutwidth of partial solutions. Additionally, we introduce a Greedy Randomized Adaptive Search Procedure (GRASP) heuristic to obtain good initial solutions. The combination of the branch-and-bound and GRASP methods results in optimal solutions or a reduced relative gap (difference between upper and lower bounds) on the instances tested. Empirical results with a collection of previously reported instances indicate that the proposed algorithm is able to solve all the small instances (up to 32 vertices) as well as some of the large instances tested (up to 158 vertices) using less than 30 minutes of CPU time. We compare the results of our method with previous lower bounds, and with the best previous linear integer formulation solved using Cplex. Both comparisons favor the proposed procedure.

1. Introduction

Let $\mathcal{G}(\mathcal{V},\mathcal{E})$ be a graph with vertex set $\mathcal{V}$ ($|\mathcal{V}|=n$) and edge set $\mathcal{E}$ ($|\mathcal{E}|=m$). A labeling or linear arrangement f, assigns the integers $\{1,2,\dots,n\}$ to the vertices of $\mathcal{G}$ in such a way that each vertex $v \in \mathcal{V}$ has a different label $f(v)$ (i.e., $f(v) \neq f(u)$ for all $u,v \in \mathcal{V}$ where $v \neq u$). The cutwidth of v, with respect to f, $CW_f(v)$, is the number of edges $(u,w) \in \mathcal{E}$ satisfying $f(u) \le f(v) < f(w)$. Note that $f(u)=f(v)$ if and only if $u=v$. Then, the cutwidth of v is computed as:

$$CW_f\ (v) = \left|\{(u,w) \in \mathcal{E} : f(u) \le f(v) < f(w)\}\right|. \tag{1}$$

Therefore, the vertex with label n has an associated cutwidth of 0. Given f, the cutwidth of $\mathcal{G}$ is defined as:

$$CW_f\ (\mathcal{G}) = \max_{v \in V} CW_f(v). \tag{2}$$

The optimum cutwidth of $\mathcal{G}$, $CW(\mathcal{G})$, is defined as the minimum $CW_f(\mathcal{G})$ value over all possible labelings. In other words, the cutwidth minimization problem consists of finding an f that minimizes $CW_f(\mathcal{G})$ over the set Π_n of all possible labelings.

$$CW(\mathcal{G}) = \min_{f \in \Pi_n} CW_f(\mathcal{G}). \tag{3}$$

Finding the optimum cutwidth is usually referred to as the Cutwidth Minimization Problem (CMP). This is an *NP-hard* problem as stated in Gavril [13] even for graphs with a maximum degree of three [19]. Practical applications of the CMP can be traced back to the early seventies. Adolphson and Hu [1] used it as the theoretical model to establish the number of channels in an optimal layout of a circuit (see also [1,20]). More recent applications of this problem include network reliability [17], automatic graph drawing [23] and information retrieval [2]. Despite of the practical applicability of the CMP, researchers on heuristic optimization have paid little attention to it. We have only found three references concerning heuristic methods for this problem. Specifically, a Simulated Annealing method [5], an Evolutionary Path Relinking [28] and, more recently, a Scatter Search procedure [24], which as far as we know, obtains the best results so far.

Figure 1.a is an example of an undirected graph with six vertices and ten edges. A labeling of this graph is depicted in Fig. 1.b, setting the vertices in a line in the labeling order as commonly represented in the cutwidth problem. In this way, since $f(A)=1$, vertex A comes first, followed by vertex D ($f(D)=2$) and so on. We represent f with the ordering (A,D,E,F,B,C), meaning that vertex A is located in the first position (label 1), vertex D is located in the second position (label 2) and so on. In Fig. 1.b, the cutwidth of each vertex is represented as a dashed line with its corresponding value at the bottom. For example, the cutwidth of vertex A is $CW_f(A)=5$, because the edges (A,D),(A,E),(A,F),(A,B) and (A,C) have an endpoint in A labeled with 1, and the other

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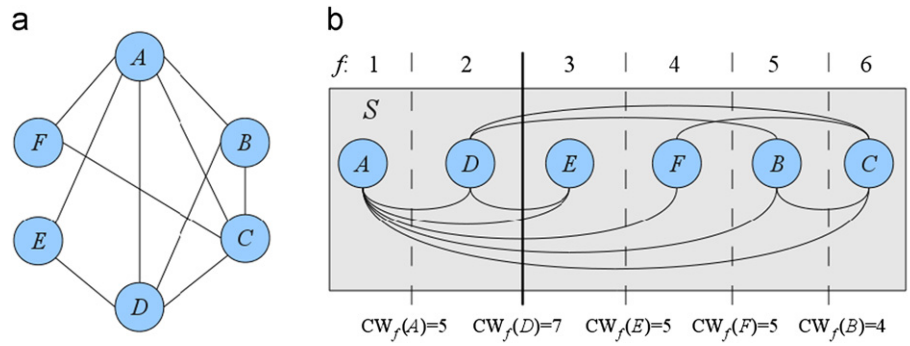

Fig. 1. (a) Graph example, (b) Cutwidth of $\mathcal{G}$ for f.

endpoint in a vertex labeled with a value larger than 1. Similarly, we can compute the cutwidth of vertex B, $CW_f(B)=4$, by counting the appropriate number of edges ((A,C),(D,C),(F,C) and (B,C)). Then, since the cutwidth of $\mathcal{G}$,$CW_f(\mathcal{G})$, is the maximum of the cutwidth of all vertices in $\mathcal{V}$, in this particular example we obtain $CW_f(\mathcal{G})=CW_f(D)=7$, represented in the figure as a bold line with the corresponding value at the bottom.

In this paper we propose a branch-and-bound algorithm for the Cutwidth Minimization Problem. It basically consists of a systematic enumeration of all its solutions (labelings) based on the definition of *partial solutions*. We review the related literature on the CMP in Section 2 and propose four new lower bounds in Section 3 that will enable us to discard a large number of solutions in the enumeration process. This latter section ends with a study of the dominance among the lower bounds. In Section 4 we study the relative dominance among nodes in the search tree. In Section 5 we introduce a heuristic based on the Greedy Randomized Search Procedure (GRASP) methodology to obtain an initial upper bound for the CMP. The reader is referred to Resende and Ribeiro [27]; Festa and Resende [9] and Festa and Resende [10] for further details concerning the GRASP methodology. In Section 6, we describe the search tree and its associated strategies for an efficient enumeration of the problem solutions, and the paper concludes with the computational experiments and the associated conclusions.

2. Previous methods, bounds and formulations

The CMP has been optimally solved for some special classes of graphs. For example, Harper [15] solved the cutwidth for hypercubes, Chung et al. [4] presented an $O(\log^{d-2} n)$ time algorithm for the cutwidth of trees with n vertices and with maximum degree d. Yannakakis [32] improved these results by giving an $O(n\log n)$ time algorithm for the same kind of graphs. In particular, for a complete t-ary tree with k-levels (heigh k), $T_{t,k}$, it holds that:

$$CW_f(T_{t,k})=\left\lceil \frac{1}{2}(k-1)(t-1)\right\rceil+1, \quad \forall\, k\geq 3. \tag{4}$$

Exact methods to obtain the optimal cutwidth of grids have been proposed in Rolim et al. [30]. Specifically, for a grid $L_{w,h}$ with width $w\geq 2$ and height $h\geq 2$, these authors proved that:

$$CW(L_{w,h})=\begin{cases} 2, & \text{if } w=h=2 \\ \min\{w+1,h+1\}, & \text{otherwise} \end{cases}. \tag{5}$$

Recently, Thilikos et al. [31] presented an algorithm to compute the cutwidth of bounded degree graphs with small treewidth in polynomial time. As far as we know, there is no previous exact method for the CMP on general graphs, and all the previous methods, as shown above, target special classes of graphs. However, we have identified four previous lower bounds and a linear integer formulation that we describe in the following subsections.

2.1. Lower bounds for the CMP

Díaz et al. [6], proposed two lower bounds for the CMP. The first one is based on fundamental cuts and the second one in spectral properties of graphs. The computation of the former is based on the well-known max-flow min-cut theorem [11], which states that the maximal flow value from an origin o to a destination d in a given graph is equal to the minimal edge cut separating o and d (called a *fundamental cut*). If we compute the value of the fundamental cut for all the possible pairs (o,d) in a given graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, the maximum of these values is a lower bound of the CMP [6] that we denote as LB_{FF}. In mathematical terms:

$$CW(\mathcal{G})\geq LB_{FF}=\max_{o,d\in\mathcal{V}}\{cut(o,d)\}, \tag{6}$$

where $cut(o,d)$ represents the size of the fundamental cut from o to d.

Considering the Laplacian matrix associated to a graph, it is possible to derive a lower bound for the CMP using its second smallest eigenvalue [16]. Given a connected graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ with $|\mathcal{V}|=n$, let λ_2 be the second smallest eigenvalue. The LB_{LM} lower bound can be computed as:

$$CW(\mathcal{G})\geq LB_{LM}=\lambda_2\frac{\lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil}{n}. \tag{7}$$

Additionally, we can derive new lower bounds by studying the relations of the CMP with other layout optimization problems. Specifically, Díaz et al., [6] presented an inequality between the CMP and the Minimum Linear Arrangement problem, MinLA [12,25], and another one between the CMP and the Edge Bisection problem, EB [12]. Given a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ with $|\mathcal{V}|=n$, and a labeling f, then:

$$LA_f(\mathcal{G})\leq n\cdot CW_f(\mathcal{G}), \quad \text{with } f\in\Pi_n \text{ and} \tag{8}$$

$$EB_f(\mathcal{G})\leq CW_f(\mathcal{G}), \quad \text{with } f\in\Pi_n. \tag{9}$$

where $LA_f(\mathcal{G})$ and $EB_f(\mathcal{G})$ are the values of the MinLA and EB objective functions, respectively. Consequently, two additional lower bounds, LB_{MinLA} and LB_{EB} can be derived:

$$CW(\mathcal{G})\geq LB_{MinLA}=\frac{LA(\mathcal{G})}{n}, \tag{10}$$

$$CW(\mathcal{G})\geq LB_{EB}=EB(\mathcal{G}). \tag{11}$$

2.2. Integer programming model

Luttamaguzi et al. [18] proposed the following CMP linear integer formulation based on the binary decision variables x_i^k, with indices $i,k\in\{1,2,\ldots,n\}$, specifying whether i is placed in position k in the ordering. This binary variable takes on value 1 if and only if i occupies the position k in the ordering; otherwise