Computers & Operations Research 39 (2012) 698–708

Contents lists available at ScienceDirect

Computers & Operations Research

journal homepage: www.elsevier.com/locate/caor

Interior point methods in DEA to determine non-zero multiplier weights

M.L. Bougnol [a,*], J.H. Dulá [b], P. Rouse [c]

[a] Jacksonville University, Jacksonville, FL 32211, USA
[b] Virginia Commonwealth University, Richmond, VA 23284, USA
[c] The University of Auckland, Auckland 1142, New Zealand

ARTICLE INFO

Available online 13 May 2011

Keywords:
Data envelopment analysis (DEA)
Healthcare pricing
Interior point methods

ABSTRACT

Multiplier weights in DEA are obtained by solving any one of several multiplier linear programs (LPs). These weights are a fundamental aspect of DEA and have many uses and interpretations including determining marginal rates of substitution. Obtaining usable values from multiplier weights can be problematic due to the way DEA LPs are formulated and solved. For example, extreme efficient DMUs generate multiple optima in standard multiplier LPs. This is also true of any efficient or virtual DMU on a face of the production possibility set with less than full dimension. Another problem arises when the LP generates optimal solutions where one or more of the multipliers are zero. An important class of interior point algorithms for LP known as "path-following" methods addresses these two issues about finding optimal multiplier weights in DEA: (1) reproducibility, that is, the optimal solution is independent of a starting point since it is generated by applying a well-defined optimization criterion; (2) non-zero multipliers, whereby the multiplier weights associated with an optimal solution for a point in the efficient frontier are never zero. In the process of exploring these methods for DEA we introduce the "multiplier generator" DEA LP formulated to provide access to all multiplier vectors for points on the efficient frontier. Our results provide prescriptions and recommendations for using path-following solvers in DEA.

Scope and purpose. This is a study on the generation of non-zero weights in DEA using interior point methods. The purpose is to generate these weights efficiently and in a manner that can be replicated independently. There is a clear demand for non-zero multiplier weights in DEA for use to price the attributes. We introduce new LP formulations specifically designed to provide access to the full set of multiplier weights at the DMU being scored.

1. Introduction

Multiplier weights are a fundamental aspect of DEA. It is what makes this methodology different from other types of productivity and performance analyses. An area of interest in DEA focuses on understanding the information content of multiplier weights. The original Charnes et al. [1] seminal work noted their use as marginal productivities for determining marginal rates of substitution.

In the case of a single input such as expenditure, multiplier weights can play a key role in apportioning expenditure and providing a top-down costing of specific outputs. Rouse [2] describes the use of multiplier weights to improve the construction of prices for a national pricing framework for the New Zealand health sector, as well as setting target costs for major output classes within health service categories such as surgical, medical, pregnancy, and childbirth.

The analyst who employs DEA for the information provided by the multiplier weights faces some well-known difficulties. Multiplier weights can be obtained directly from the optimal solution of one of many multiplier DEA LP formulations. For the important case of extreme efficient DMUs, these LPs have alternate optima and, because of the preponderance of anchor points[1] in real DEA data (see [3]), it is not uncommon to find zeroes for the values of some of the multipliers when scoring extreme efficient DMUs.

From a managerial perspective, there are always numerous measures that are potential candidates for measuring performance. The need for parsimony in the selection of model inputs and outputs conveys a message that these are regarded as especially important to the appraisal process. Zero multiplier weights for an input or output sends a contradictory message to a manager as it

* Corresponding author. Tel.: +1 904 256 7432.
E-mail addresses: mbougno@ju.edu (M.L. Bougnol), jdula@vcu.edu (J.H. Dulá), p.rouse@auckland.ac.nz (P. Rouse).

[1] An anchor point is an extreme efficient DMU which also belongs to a non-efficient "free disposability" face.

0305-0548/$ - see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.cor.2011.05.006

implies that these are not regarded as important for the evaluation. This is particularly disturbing when a DMU is assessed as technically efficient using less than the full set of model variables. This makes it difficult to distinguish between DMUs where (i) inclusion of the zero weighted input or output would disadvantage its efficiency score and (ii) an input or output is assigned a zero weighting purely because of the way the solution algorithm terminates. The imposition of weight constraints can help to distinguish between these two scenarios but they come at a cost as well as imposing greater complexity on the model.

Conceptually, a zero weighted input implies that the output can be produced without it. Alternatively a zero weighted output implies that this output does not need to be produced. This is unlikely and again it is difficult to distinguish between the scenario where forcing the DMU to include the input or output would disadvantage its efficiency score versus the scenario where the zero weight is a function of the solution algorithm. In the first scenario, the marginal rate of substitution is distorted because the zero weighted input is treated as a free good. A similar interpretation applies to zero weighted outputs. Perhaps the worst scenario is under variable returns to scale where the simplex solution can show the entire efficiency score being obtained from the scale factor with zero weights on all the outputs or inputs. Naturally, there is always an alternative optimum that will obtain strictly positive scores on some of the inputs or outputs but again this confuses managers and confounds calculations of marginal rates of substitution and virtual weights.

Alternate optima and zero multipliers represent complications for the use of these optimal solutions as multiplier weights. Without a rule or criterion for selecting a solution from among alternate optima, these solutions are essentially arbitrary. This, in turns, affects reproducibility. Values of zero for multiplier weights make them unusable for the purpose of calculating marginal rates of substitution.

These issues have been addressed before. Charnes et al. [4] propose a procedure for generating non-zero multipliers for the Constant Returns to Scale (CRS) model (see [4, Appendix B, p. 234]). Their approach starts with an optimal multiplier solution with one or more zeroes and proceeds to solve special LPs to generate solutions specifically to replace the zeroes with non-zero values. These solutions are then combined to obtain one non-zero multiplier set. The final solution depends on the initial solution and its distribution of zeroes. In the paper by Cooper et al. [5], the problem with arbitrariness and zero multipliers is resolved by formulating and solving a pair of mixed integer LPs. This approach is able to produce solutions with non-zero multipliers but since these solutions correspond to the normal of a facet of the production possibility set, they could be shared by other extreme efficient DMUs; that is, the optimal multiplier solutions provided by Cooper et al. [5] may not be specific to the extreme efficient DMU that generated them.

In this paper, we explore a class of interior point methods (IPMs) as a way to generate non-zero multiplier solutions that are specific to any point on the efficient frontier and satisfy an optimization criterion. As part of the development, we study the relevant aspects of the geometry of the production possibility set and introduce a DEA "multiplier generator" LP formulated to include all possible multiplier vectors at points on the efficient frontier. This is important because standard oriented formulations such as the input and output-oriented LPs of Banker et al. [6] may make some multiplier solutions unavailable. We also treat theoretical and practical aspects behind interior point methods, especially as they relate to generating useful multiplier solutions for DEA. All the results are illustrated using a small example and, at the end, testing is performed on actual data from the healthcare industry.

2. DEA and linear programming

LP formulations and their solutions have always been an integral part of DEA. LPs to identify efficient entities appear with the first paper introducing DEA by Charnes et al. [1]. Since then, several DEA LP formulations have been proposed and many more are possible depending on benchmarking objectives or measures of efficiency required.

The DEA analyst typically relies on familiar tools to solve the LPs such as what might be available in a commercial spreadsheet or perhaps something more specialized. (For a comprehensive study of DEA software, refer to Barr [7].) With the notable exception of Scheel's EMS [8], chances are that the simplex algorithm will solve the LPs.

The simplex algorithm is not always the ideal algorithm for solving DEA LPs. The standard DEA envelopment LP such as the one originally proposed by Charnes et al. [1] or the additive model of Charnes et al. [9], for example, induce degeneracy by duplicating the data for one of the DMUs in the right-hand side of the LP. Problems with simplex cycling due to degeneracy in DEA have been reported [10–12]. Simplex generated solutions to some DEA LPs do not always provide necessary and sufficient conditions to classify DMUs as efficient or inefficient. Finally, simplex solutions can be problematic when the purpose of the LP is to obtain prices and rates of substitution for DMUs' attributes.

The simplex algorithm is not the only choice for solving LPs. An efficient algorithm for LP was introduced by Karmarkar [13] in 1984. This algorithm gave rise to a class of procedures known as path-following interior point methods (IPM). Unlike the simplex algorithm which moves along the boundary of the feasible region, path-following IPMs attempt to track a specific interior "central path" to optimality. IPM implementations differ on how closely the central path is actually approximated. The end of the central path is an extreme point when the optimal solution is unique and both the IPM and the simplex algorithm find it. Unlike the simplex, which always terminates at an extreme point, the point at the end of the central path is in the interior of the optimal face which is not an extreme point if the face has one or more dimensions. Such optimal solutions satisfy the strict complementary slackness condition (SCSC) that exactly one of the members in a complementary pair is zero [14].

Theoretical path-following IPMs to solve DEA LPs promise tangible advantages:

1. Issues of cycling due to degeneracy do not affect IPMs. This has an immediate benefit for DEA computations.
2. IPMs find the optimal solution to an LP which is in a specific, well-defined, central location on the optimal face: the "analytic center." The analytic center satisfies an optimization criterion and is guaranteed to satisfy SCSC.
3. SCSC solutions provide necessary and sufficient conditions for DMU classification. IPMs obviate the use of the problematic non-Archimedean LP formulation or the practice of solving a second LP whenever the first solution does not lead to a conclusive classification.
4. SCSC solutions for multiplier LPs when scoring points on the efficient frontier of the production possibility set are guaranteed to generate non-zero multipliers.

The last of these properties is the one we wish to explore in this work. We will investigate the theory and practice behind generating non-zero optimal solutions to the multiplier LP formulations using path-following IPMs when used to score extreme efficient DMUs and other efficient DMUs on lower dimensional faces. In the next section, we formalize this result specifically for the purpose of solving DEA LPs.