Computers & Operations Research 39 (2012) 127–138

Contents lists available at ScienceDirect

Computers & Operations Research

journal homepage: www.elsevier.com/locate/caor

A self-adaptive gradient projection algorithm for the nonadditive traffic equilibrium problem

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ARTICLE INFO

Available online 25 February 2011

Keywords:
Traffic equilibrium problem
Nonadditive route cost
Gradient projection algorithm
Self-adaptive scheme

ABSTRACT

Gradient projection (*GP*) algorithm has been shown as an efficient algorithm for solving the traditional traffic equilibrium problem with additive route costs. Recently, *GP* has been extended to solve the nonadditive traffic equilibrium problem (*NaTEP*), in which the cost incurred on each route is not just a simple sum of the link costs on that route. However, choosing an appropriate stepsize, which is not known *a priori*, is a critical issue in *GP* for solving the *NaTEP*. Inappropriate selection of the stepsize can significantly increase the computational burden, or even deteriorate the convergence. In this paper, a self-adaptive gradient projection (*SAGP*) algorithm is proposed. The self-adaptive scheme has the ability to automatically adjust the stepsize according to the information derived from previous iterations. Furthermore, the *SAGP* algorithm still retains the efficient flow update strategy that only requires a simple projection onto the nonnegative orthant. Numerical results are also provided to illustrate the efficiency and robustness of the proposed algorithm.

1. Introduction

A basic assumption of the traditional traffic equilibrium model is additivity (i.e., the route cost is simply the sum of the costs on the links that constitute that route). The biggest advantage of the additivity assumption is that it allows the route-flow variables to be removed from the objective function of the convex mathematical programming (*MP*) formulation (in the case of symmetric link travel time functions) or from the inequality of the variational inequality (*VI*) formulation (in the case of asymmetric link travel time functions). Thus, the corresponding traffic equilibrium problem can be solved without the need to store routes despite that the route-flow variables remain in the constraint set. This is a significant benefit when one needs to solve large-scale problems in transportation networks.

However, as pointed out by Gabriel and Bernstein [1], the additivity assumption is not appropriate in many real life situations, where the additive route cost structure is inadequate for addressing factors affecting a variety of transportation policies, such as nonlinear value of travel times, route-specific tolls, and emissions fees. A few studies have been performed by using different nonadditive route cost structures, such as the route-specific travel costs [2,19], bi-criteria nonlinear route costs with elastic demand [3], length-based and congestion-based commonality factors used in the C-logit stochastic user equilibrium model [4], entry-exit based toll charges [5,6], and feeder bus systems [7]. Recently, the nonadditive route costs have been applied to model risk-averse behavior in the route choice decision process [8–12]. Some formulations and properties of the nonadditive traffic equilibrium problems (*NaTEP*) were also explored, such as the nonlinear time/money relation [13], uniqueness and convexity of the bi-criteria traffic equilibrium problem [14], and the monotonicity of *NaTEP* formulated as a monotone mixed complementarity problem [15]. Furthermore, Altman and Wynter [16] discussed the nonadditive cost structures in both transportation and telecommunication networks.

Under the nonadditivity assumption, the corresponding traffic equilibrium problem has to be formulated in the route-flow space, and thus cannot be solved using the traditional link-based algorithms, such as the Frank–Wolfe algorithm (see [17] for the details of the algorithm). Furthermore, as indicated by Bernstein and Gabriel [18], the diagonalization methods also do not work well on the nonadditive problem, since the diagonalized subproblems are poor approximations of the true problem.

Different formulations and solution approaches have been presented for solving various nonadditive traffic equilibrium problems. For example, Bernstein and Gabriel [18] presented a non-smooth equation/sequential quadratic programming (NE/SQP) method for solving the *NaTEP* with elastic demands. The method is based on first transforming the nonlinear

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0305-0548/$ - see front matter Published by Elsevier Ltd.
doi:10.1016/j.cor.2011.02.018

complementarity problem (*NCP*) formulation of the *NaTEP* into a set of non-smooth equations and then finding the zero point of a non-smooth, non-convex optimization problem. Lo and Chen [19] used a new gap function proposed by Fischer [20] to convert the *NCP* formulation to an equivalent unconstrained optimization problem, which is smooth and convex. The unconstrained nature makes available a large number of already developed solution algorithms, such as the Newton method, Quasi-Newton method, Gradient method, etc. [21]. Lo and Chen [22] provided an alternate formulation via a smooth gap function using both route flows and O–D costs as the decision variables. Chen et al. [2] suggested a self-adaptive projection and contraction (*PC*) algorithm for solving a monotone *VI* formulation as a special case of the *NaTEP* with elastic demands. Han and Lo [23] also proposed a descent method for the co-coercive *VI* formulation to solve the *NaTEP* with elastic demand. The advantage of the last two algorithms is its simplicity in numerical implementations, since both only need some function evaluations of the mapping and a few trivial projection operations on the nonnegative orthant.

On the other hand, the gradient projection (*GP*) algorithm has been shown as a successful route-based algorithm for solving the traditional traffic equilibrium problem with additive route costs [24,25]. Under an ingenious approach that utilizes the special structure of the traffic equilibrium problem (see the implementation by Jayakrishnan et al. [24] and Chen et al. [25]), *GP* only needs to perform a simple projection on the nonnegative orthant in each iteration; therefore, the required computational effort is modest. Previous results reported by Jayakrishnan et al. [24] and Chen et al. [25] on the *GP* algorithm adopt the diagonal inverse Hessian approximation as a scaling matrix, assume a unity stepsize in each iteration, and use the "*one-at-a-time*" flow update strategy to equilibrate route flows one origin–destination (O–D) pair at a time. Though a near-optimal solution (e.g., 0.001 as the stopping criterion) can be achieved quickly, *GP* may have difficulty in obtaining the optimal solution (i.e., very accurate solution). This is partly due to the unity stepsize assumption purposely designed in *GP* to avoid expensive line searches. Attracted by the efficiency and simplicity of the *GP* algorithm, Scott and Bernstein [26] extended it for solving the *NaTEP* by using the '*all-at-once*' flow update strategy and a modified scaling matrix to reflect the nonadditive route costs. However, the problem of choosing an appropriate stepsize is a critical issue in *GP* for solving the *NaTEP* due to the complex route cost structure. The results reported in [26] were mixed. For the Sioux Falls network, a "*trial-and-error*" approach of choosing a fixed stepsize was used to ensure convergence.

Thus, our study is motivated to develop a robust stepsize scheme in *GP* for solving the *NaTEP*. In this paper, a self-adaptive gradient projection (*SAGP*) algorithm is provided. The self-adaptive scheme has been successfully embedded in the original Goldstein–Levitin–Polyak (*GLP*) projection algorithm by Han and Sun [27] and demonstrated by Zhou and Chen [28] for solving the asymmetric traffic equilibrium problem. It has the ability to automatically adjust the stepsize according to the information derived from previous iterations. Thus, it is not necessary to use the "*trial-and-error*" approach as suggested by Scott and Bernstein [26] to select a suitable fixed stepsize. The self-adaptive scheme can significantly enhance the robustness and efficiency of the algorithm. Furthermore, the *SAGP* algorithm still retains the simple flow update strategy (i.e., simple projection on the nonnegative orthant), thus avoiding the need to solve convex quadratic programs in the original *GLP* projection algorithm.

The remainder of the paper is organized as follows: a general *VI* formulation of the *NaTEP* is given in Section 2; Section 3 discusses the gradient projection algorithm; Section 4 presents the self-adaptive gradient projection (*SAGP*) algorithm as well as its convergence; numerical results are provided in Section 5 to illustrate the efficiency and robustness of the *SAGP* algorithm; finally, conclusions are summarized and some future researches are suggested in Section 6.

2. Formulation of the nonadditive traffic equilibrium problem

Throughout this study, we assume the origin–destination (O–D) travel demands are given and fixed. Consider a strongly connected network $[N, A]$, where N and A denote the sets of nodes and links, respectively. Let R and S denote a subset of N for which travel demand q^{rs} is generated from origin $r \in R$ to destination $s \in S$. The assumption of a strongly connected network guarantees that there exists at least one route from every O–D pair with positive travel demand. Let f_p^{rs} denote the flow on route $p \in P^{rs}$, where P^{rs} is a set of routes from origin r to destination s. Let $\Delta = [\delta_{pa}^{rs}]$ denote the route–link incidence matrix, where $\delta_{pa}^{rs} = 1$ if route p from origin r to destination s uses link a, and 0, otherwise. Then, we have the following relationships:

$$q^{rs} = \sum_{p \in P^{rs}} f_p^{rs}, \quad \forall r \in R, \ s \in S, \tag{1}$$

$$v_a = \sum_{r \in R} \sum_{s \in S} \sum_{p \in P^{rs}} f_p^{rs} \delta_{pa}^{rs}, \quad \forall a \in A, \tag{2}$$

$$f_p^{rs} \geq 0, \quad \forall p \in P^{rs}, \ r \in R, \ s \in S, \tag{3}$$

where (1) is the travel demand conservation constraint; (2) is a definitional constraint that sums up all route flows that pass through a given link a; and (3) is a non-negativity constraint on route flows.

Under the symmetric link cost and additive route cost assumptions, the traditional traffic equilibrium model can be formulated as a convex mathematical program and solved by a link-based traffic assignment algorithm (e.g., the Frank–Wolfe algorithm). However, under the nonadditive route cost structure (i.e., not only that the additivity assumption does not hold, the symmetry assumption is also not satisfied), it is therefore necessary to formulate the problem using route-flow variables and solve it with a route-based traffic assignment algorithm. A general nonadditive route cost function can be written as follows [1]:

$$\eta_p^{rs} = \gamma_p^{rs} + \sum_{a \in A} \rho \delta_{pa}^{rs} t_a + g_p\left(\sum_{a \in A} \delta_{pa}^{rs} t_a\right), \quad \forall p \in P^{rs}, \ r \in R, \ s \in S, \tag{4}$$

where γ_p^{rs} denotes the financial cost (such as toll) specific to route p between origin r and destination s, ρ is the operating cost per unit travel time (e.g., fuel consumption, vehicle rental), and g_p is a function describing the value of time for route p, which could be nonlinear. The second and third terms transfer travel times into an equivalent amount of money consistent with the first term. Typically, the general route cost function η_p^{rs} and the link travel time function t_a are assumed to be positive and continuous; the operating cost factor ρ is positive; the route-specific cost γ_p^{rs} and the valuation of time function $g_p(\cdot)$ are continuous and nonnegative.

Let $\boldsymbol{\eta}$ denote the route-cost vector $(\ldots,\eta_p^{rs},\ldots)^T$, π^{rs} denote the minimal cost between O–D pair (r, s), and $\mathbf{f}$ denote the route-flow vector $(\ldots,f_p^{rs},\ldots)^T$. The traffic equilibrium problem is to find the traffic-flow pattern by allocating the O–D demands to the network such that all used routes between each O–D pair have equal and minimum travel cost, and no unused route has a lower