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Hybrid-LP: Finding advanced starting points for simplex, and pivoting LP methods ☆

Camelia Al-Najjar *, Behnam Malakooti

Systems Engineering Program, Department of Electrical Engineering & Computer Science, Case School of Engineering, Case Western Reserve University, 10900 Euclid Ave, Cleveland, OH 44106, USA

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ABSTRACT

The simplex method has proven its efficiency in practice for linear programming (LP) problems of various types and sizes. However, its theoretical worst-case complexity in addition to its poor performance for very large-scale LP problems has driven researchers to develop alternative methods for LP problems. In this paper, we develop the hybrid-LP; a two-phase approach for solving LP problems. Rather than following a path of extreme points on the boundary of the feasible region as in the simplex method, the first phase of the hybrid-LP moves through the interior of the feasible region to obtain an improved and advanced initial basic feasible solution (BFS). Then, in the second phase simplex or other LP methods can be used to find the optimal solution.

Since the introduction of polynomial-time methods for LP, a considerable amount of research has focused on interior-point methods for solving large-scale LP problems. Although fewer iterations are needed for interior-point methods to converge to a solution, the iterations are computationally intensive. Our approach is a hybrid method that uses a computationally efficient pivot to move in the interior of the feasible region in its first phase. This single iteration is able to bypassing several extreme points to an improved BFS, which can then be used as a starting point in any LP method in the second phase of the method. Our approach can also be modified to perform a number of interior pivots in the first phase based on the trade-off between the number of iterations and the running time.

The hybrid-LP uses an efficient pivoting iteration which is computationally comparable to the standard simplex iteration. Another feature is adaptability in finding the advanced starting point by avoiding the boundaries of the feasible region. In addition, the hybrid-LP has the ability to start from a feasible point which may not be a BFS. Our computational experiments demonstrate that the hybrid-LP reduces both the number of iterations and the running time compared to the simplex method on a wide range of test problems.

1. Introduction

Ever since the simplex method emerged for solving linear programming (LP) problems, its efficiency was demonstrated in practice extensively despite its theoretical worst-case performance. After Klee and Minty [12] demonstrated the worst-case exponential performance of simplex on a certain LP problem structure, interest was sparked in alternative methods.

After the introduction of polynomial-time interior-point methods, the majority of research in LP focused in this area. Interior-point methods have been shown to be more efficient than the simplex method in solving sparse large-scale LP problems. However, the simplex method remains powerful for small- and medium-size problems, branch and bound methods for integer programming, sensitivity analysis, handling changes in the formulation (such as addition of constraints or decision variables), and taking advantage of prior knowledge or an estimate of the solution. Solutions to LP problems by interior-point methods require few iterations; however, interior-point iterations can be

☆ Statement of scope and purpose: The focus of this paper is to develop the hybrid-LP: a method for solving linear programming (LP) problems that combines both interior and boundary paths in the search for the optimal point. In the simplex method, the standard selection of the initial point does not take into consideration the value of the objective function or the location of the optimal. Since the simplex methods follow a path of extreme points on the boundary of the feasible regions, the number of simplex iterations required to obtain an optimal point may be exponentially high as demonstrated by Klee and Minty [12]. The hybrid-LP uses an interior direction to pass through the feasible region to an improved basic feasible solution (BFS) upon which the simplex method can be used to proceed to the optimal with a reduced number of iterations. Our experimental results show the hybrid-LP reduces both the number of iterations and the running time compared to the simplex method. We predict that an optimized implementation can further improve the computational efficiency of our method.

* Corresponding author.
E-mail addresses: caa20@case.edu (C. Al-Najjar), bxm4@case.edu (B. Malakooti).

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up to a thousand times more computationally intensive than a simplex iteration [20].

The average performance of the simplex method has been shown to be polynomial by Borgwardt et al. [1]. Practical implementations for linear optimization commonly use the simplex method to solve LP problems of various types. In addition, certain heuristics are applied which allow the simplex method to avoid its worst-case exponential bound. The reader is referred to Ref. [20] for a review of the strengths and weaknesses of simplex vs. interior-point methods.

Still, there has been some research focus on variations of the simplex method and methods which follow the simplex framework. Few different pivot rules have been developed for the simplex method; however, it was shown that most have the same theoretical worst-case exponential complexity as the standard pivot rule (a review of which is in Ref. [14]). Another approach is followed by the methods of Paparizzo [17], Paparizzo et al. [18], and Chen et al. [3], which are pivoting methods that pivot on basic solutions which are infeasible; these methods are known as exterior-point simplex.

A number of other methods were developed which followed an interior search direction in the feasible region within the simplex framework; these methods are known as external pivoting since they pivot on artificial variables which are combinations of existing variables. External pivoting was first introduced by Eiselt and Sandblom [6]. Subsequent modifications and experimentation in external pivoting are presented by Mitra et al [15], Fathi and Murty [9], and Eiselt and Sandblom [7].

A different approach is taken by Stojkovic and Stanimirovic [19], Junior and Lins [11], and Luh and Tsaih [13], who rather than improve on the simplex algorithm itself, developed a method to select a better starting point for the simplex method which reduces the number of simplex iterations needed. The first two methods are based on estimating an optimal (or near-optimal) basis by finding constraints which intersect the gradient plane at minimal angles. The third method develops a search direction which is a combination of the gradient direction and an internal pointing direction with respect to the polyhedral forming the feasible region. The authors Chaderjian and Gao [2] and Hu [10] offer modifications and simplifications of Refs. [13] and [11], respectively.

In this paper, we present the hybrid-LP: a two-step method in which the first step follows an interior search direction to an improved basic feasible point. The interior direction is derived from the reduced gradient in such a way that it avoids the boundary. Then, the second step uses the simplex method to reach the optimal point. Experiments on randomly generated test problems and on NETLIB test problems [16] show promising results in reducing the number of iterations and the running time compared to the simplex method.

The method includes more flexibility in determining the search direction than the external pivoting methods [15,9,7] or the improved starting point [13,2]. This flexibility is necessary to avoid the boundaries and achieve better improvements toward the optimal point as illustrated in Section 2.

The algorithm uses a computationally efficient pivot-based operation similar to the simplex iteration except more than one variable (or column) is involved in the pivot. The simplex framework is maintained; thus sensitivity analysis, varying coefficients, variables, and constraints, as well as warm-start are all possible in this method. As a result, the hybrid-LP competes with the simplex method in its domain of small to medium-sized LPs. A comparison to interior-point methods is not necessary within the scope of this paper and is left for future work and experimentation with sparse large-scale LPs.

This paper is organized as follows. In Section 2, we explain the motivation for generating an improved initial point for the simplex method and the main idea behind our method. In Section 3, the steps of the algorithm are presented. Our experimental results are reported in Section 4. Section 5 consists of discussion of the implementation parameters, a comparison to similar methods, and directions for future work. Our conclusions are presented in Section 6.

2. Motivation

Since the feasible region in LP is a convex set, our interior search direction method developed in this paper can ideally obtain the optimal solution in one iteration. The difficulty lies in finding a good search direction that leads to the optimal point or close to it. Rather than heading in a direct path towards the optimal, the simplex method moves in a path of extreme points on the boundary of the feasible region. Despite this fact, it is very efficient in practice for various LP problem sizes and types. However as the problem gets larger, the number of extreme points is increasing exponentially, and an exponential number of simplex pivots may be required. Therefore, there may be some room for improvement to the amount of computational effort needed to solve LP problems by the simplex method.

The first phase of the hybrid-LP method is based on a similar pivot operation to that of the simplex method; however, the pivot is performed over a direction that is a combination of several non-basic variables, rather than a single non-basic variable as is the case with the simplex method. This combined direction is an interior direction with respect to the feasible region; therefore, the direction is able to bypass several extreme points and head to a point closer to the optimal. Using an interior direction also allows larger improvements in the objective function than that of a move on the boundary of the feasible region. The selection of the direction of improvement is flexible and can be adapted to each problem. A reduction process is then used to find an improved basic feasible solution, and the simplex method is started from that point as the second phase of the hybrid-LP method.

Consider an LP problem whose feasible region is illustrated in Fig. 1. Suppose the feasible region is bounded by one million linear constraints which, when scaled down, appear as a curve in Fig. 1. Suppose the optimal point is in the middle of this curve and the initial BFS for the simplex method is point (1, 0). In order to find the optimal solution, the simplex method must iterate about a half million times; as in Fig. 1a.

In addition, other gradient methods may also experience the same problem illustrated in Fig. 1. At the point (1, 0), the gradient direction points to the exterior of the feasible region, therefore, feasible direction methods that use variants of the gradient direction, conjugate gradient, or the reduced gradient generate directions also pointing along the boundary. In this case, the expected path for these methods will be the same as that of the simplex method requiring about a half million iterations.

On the other hand, our method overcomes this problem by including flexibility in choosing the interior direction. The direction is selected such that it combines both increasing and decreasing directions with respect to the objective function. As a result, the direction does not follow the boundary, but at the same time is guaranteed to be an improving direction. Since the objective function is linear, the best point on the line formed by the search direction is the far-end of the line at the boundary of the feasible region. Then from this boundary point, a basic feasible solution (BFS) is generated. This BFS is used as the starting point for the simplex method in phase two of the method. Fig. 1a illustrates the path taken by the simplex method to the optimal point and Fig. 1b shows the hybrid-LP method. It can be seen from the illustration the potential savings in the number of iterations.