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Fourier analysis of semi-discrete and space–time stabilized methods for the advective–diffusive–reactive equation: I. SUPG

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Abstract

The goal of this paper is twofold. One the one side, the most common transient Galerkin and SUPG methods are analyzed for the one-dimensional advection–diffusion–reaction equation. The methods analyzed include semi-discrete, time-discontinuous space–time stabilized finite element methods and several predictor multi-corrector versions of them. On the other hand, in the framework of explicit predictor multi-corrector algorithms a novel treatment of the source terms is proposed and analyzed. The technique consists of the diagonally implicit treatment of the negative or dissipative source terms. This technique increases dramatically the phase and damping accuracy of classical explicit methods and, at the same time, removes the source terms from stability considerations for low viscosity flows, thus, leading to very economic procedures.

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1. Introduction

In fluid transport problems, there are many transcendental phenomena which are modeled with source terms. Just to name some, we can mention the evolution of Reynolds averaged turbulence quantities, flows involving chemical reactions, incompressible flows with buoyancy source terms driven by the Boussinesq

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model, friction and infiltration terms in the shallow water equations, acoustic propagation problems, and so on.

Compared to the simple advection–diffusion problem, source terms allow a much richer spectrum of solutions, including wave propagation phenomena and exponentially growing/decaying solutions. Thus, their presence implies an added difficulty from bare advective–diffusive problems.

A good model equation for flows with large source terms is the one-dimensional advection–diffusion–reaction equation. Although methods for the advection–diffusion equation have been extendedly studied, a thorough analysis of methods including source terms is missing. Therefore, in this paper we are going to analyze the stability and accuracy of several non-stationary finite element methods applied to the one-dimensional advection–diffusion–reaction equation.

Two types of transient finite element techniques will be studied in this paper: semi-discrete methods and discontinuous-in-time space–time methods. Within semi-discrete methods the forward Euler, backward Euler and the trapezoidal rule will be considered and for space–time methods, the constant-in-time and the linear-in-time elements.

It is well known that Galerkin methods do not have enough stability for convective or reactive dominated flows. This lack of stability manifests itself in wild oscillations that can completely pollute the solution in the presence of sharp boundary or internal layers, which are due to boundary conditions or discontinuities of the model equation parameters. The approaches to cure this problem are many and varied.

Some authors stabilize the Galerkin method by choosing the right boundary conditions and the location of the first node within the layers (see [3,6,12] for examples of how this technique works). This approach is that of those who do not want to remove the wiggles [11] because they are manifesting that the mesh is too coarse there.

Other methodologies consist of enriching the Galerkin finite element spaces with bubble functions (functions that are zero along all the element boundary). The literature on this topic is very rich, so as an example see [1,2,4,7] and references therein.

The approach pursued in this paper is that of stabilized methods. In particular, we are going to analyze SUPG methods [5,23,24] and leave the analysis of SGS methods [21,26] for the sequel [17]. We will not analyze GLS methods [22] because in [14,19] it was shown that these methods need a much better tuned parameter than SUPG and SGS to achieve accuracy along all the spectrum of source terms.

The second goal of this paper is to present and analyze a novel idea for the explicit integration of the above methods. There are two classes of time stepping techniques: implicit methods and explicit methods. Implicit methods require at each iteration the inversion of a matrix. Generally, they have the unconditional stability property, that is, the methods are stable for any time step size. The other class of methods are explicit methods, which do not require any matrix inversion. Thus, they give rise to fast numerical algorithms, which are applied when the time step is small, but are conditionally stable. That is, there is a maximum limit of the time step size due to stability considerations. All these methods can be casted in the form of predictor multi-corrector algorithms [25].

Moreover, finite elements do not naturally yield truly explicit methods because the consistent mass matrix is not diagonal. Truly explicit finite element methods require an additional action, called mass lumping. The various lumping techniques available [10,12,20,30] have been basically applied up until now to the mass matrix.

Furthermore, when source terms are taken into account, unconditional stability demands from the dissipative source terms an implicit treatment, which destroys the nature of explicit methods. Where as if they are treated explicitly, then, they induce a strong restriction on the time step size due stability considerations.

Thus, we propose in this paper a diagonally implicit treatment of the dissipative source terms, via a row sum lumping technique, which not only increases the stability of the algorithm, but also improves considerably the accuracy of explicit methods. It will be shown, that this trick removes the source terms from sta-