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An efficient ensemble algorithm for numerical approximation of stochastic Stokes–Darcy equations

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Abstract

We propose and analyze an efficient ensemble algorithm for fast computation of multiple realizations of the stochastic Stokes–Darcy model with a random hydraulic conductivity tensor. The algorithm results in a common coefficient matrix for all realizations at each time step making solving the linear systems much less expensive while maintaining comparable accuracy to traditional methods that compute each realization separately. Moreover, it decouples the Stokes–Darcy system into two smaller sub-physics problems, which reduces the size of the linear systems and allows parallel computation of the two sub-physics problems. We prove the ensemble method is long time stable and first-order in time convergent under a time-step condition and two parameter conditions. Numerical examples are presented to support the theoretical results and illustrate the application of the algorithm.

Keywords: Stokes–Darcy equations; Uncertainty quantification; Ensemble algorithm; Finite element method; Partitioned method

1. Introduction

Many engineering and geological applications require effective simulations of the coupling of groundwater flows (in porous media) and surface flows. Accurate simulations are usually not feasible due to the fact it is physically impossible to know the exact parameter values, e.g., the hydraulic conductivity tensor, at every point in the domain as the realistic domains are of large scale and natural randomness occur at small scales. Consequently, these uncertainties must be taken into account to obtain meaningful results. The usual way is to model the parameter of interest as a stochastic function that is determined by an underlying random field with an prescribed (usually experimentally determined) covariance structure, and then recast the original deterministic system as a stochastic system. As a result, numerical approximations that involve repeated sampling and simulations pose great challenges on the computer resources and capability. A recently developed ensemble algorithm was devoted to address this issue. Jiang and Layton [1] studied an efficient ensemble algorithm for solving multiple realizations of evolutionary Navier–Stokes equations. The algorithm results in a common coefficient matrix for all realizations corresponding to different initial

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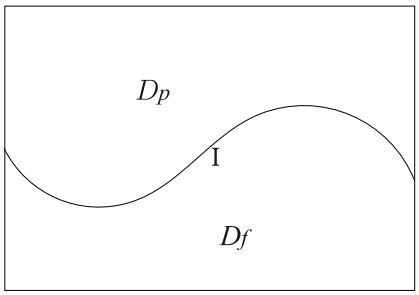

Fig. 1.1. A sketch of the porous median domain D_p, fluid domain D_f, and the interface I.

conditions or body forces, and thus efficient direct or iterative solvers can be used to reduce both required storage and computational time. This algorithm has been extensively tested and shown to be able to significantly reduce the computational cost, [2–6]. Herein we follow the same idea and develop an efficient ensemble algorithm for simulating the coupling of groundwater flows and surface flows.

In this report, we consider a linear Stokes–Darcy model for the coupling of the surface and porous media flows, where the Stokes equations describe the incompressible surface fluid flow and the Darcy model describes the groundwater flow in porous media. For derivation and more detailed discussions of the Stokes–Darcy model, see [7–13]. Let D_f denote the surface fluid flow region and D_p the porous media flow region, where $D_f, D_p \subset R^d (d = 2, 3)$ are both open, bounded domains. These two domains lie across an interface, I, from each other, and $D_f \cap D_p = \emptyset$, $\bar{D}_f \cap \bar{D}_p = I$, see Fig. 1.1.

The Stokes–Darcy model is: Find fluid velocity $u(x,t)$, fluid pressure $p(x,t)$, and hydraulic head $\phi(x,t)$ that satisfy

$$\begin{aligned}
&u_t - \nu\Delta u + \nabla p = f_f(x,t), \nabla\cdot u = 0, \quad \text{in } D_f,\\
&S_0\phi_t - \nabla\cdot(\mathcal{K}(x)\nabla\phi) = f_p(x,t), \text{in } D_p,\\
&\phi(x,0) = \phi_0(x), \text{ in } D_p \text{ and } u(x,0) = u_0(x), \text{ in } D_f,\\
&\phi(x,t) = 0, \text{ in } \partial D_p \setminus I \text{ and } u(x,t) = 0, \text{ in } \partial D_f \setminus I.
\end{aligned} \tag{1.1}$$

Let $\widehat{n}_{f/p}$ denote the outward unit normal vector on I associated with $D_{f/p}$, where $\widehat{n}_f = -\widehat{n}_p$. The coupling conditions across I are conservation of mass, balance of forces and the Beavers–Joseph–Saffman condition on the tangential velocity:

$$\begin{aligned}
&u\cdot\widehat{n}_f - \mathcal{K}\nabla\phi\cdot\widehat{n}_p = 0 \text{ and } p - \nu\,\widehat{n}_f\cdot\nabla u\cdot\widehat{n}_f = g\phi \text{ on } I,\\
&-\nu\,\nabla u\cdot\widehat{n}_f = \frac{\alpha_{\mathrm{BJS}}}{\sqrt{\widehat{\tau}_i\cdot\mathcal{K}\widehat{\tau}_i}}u\cdot\widehat{\tau}_i \quad \text{on } I, \text{ for any tangential vector } \widehat{\tau}_i \text{ on } I,
\end{aligned}$$

see [14–16]. Here, g, $\mathcal{K}$, ν and S_0 are the gravitational acceleration constant, hydraulic conductivity tensor, kinematic viscosity and specific mass storativity coefficient, respectively, which are all positive. $\mathcal{K}$ is assumed to be symmetric positive definite (SPD).

In simulations of porous media flows, the major difficulty is the determination of the hydraulic conductivity tensor $\mathcal{K}$. In the simplest case of isotropic homogeneous media, the hydraulic conductivity tensor is diagonal and constant. But in most geophysical and engineering applications, the media are usually randomly heterogeneous, and each component $k_{ij}(x,w)$ of the hydraulic conductivity tensor is a random function that depends on spatial coordinates. Then the problem becomes solving a stochastic PDE system instead of a deterministic PDE system and the goal of mathematical analysis and computer simulations is the prediction of statistical moments of the solution, such as the mean and variance. The most popular approach in solving a PDE system with random inputs is the Monte Carlo method, which is easy to implement and allows the use of existing deterministic codes. The main disadvantage of the Monte Carlo method is its very slow convergence rate $1/\sqrt{J}$, which inevitably requires computation of a large number of realizations to obtain useful statistical information from the solutions. Other ensemble-based methods have been devised to produce faster convergence rates and reduce numerical efforts including multilevel Monte Carlo method [17], quasi-Monte Carlo sequences [18], Latin hypercube sampling [19], centroidal Voronoi