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Accurate finite element modeling of acoustic waves

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ABSTRACT

In the paper we suggest an accurate finite element approach for the modeling of acoustic waves under a suddenly applied load. We consider the standard linear elements and the linear elements with reduced dispersion for the space discretization as well as the explicit central-difference method for time integration. The analytical study of the numerical dispersion shows that the most accurate results can be obtained with the time increments close to the stability limit. However, even in this case and the use of the linear elements with reduced dispersion, mesh refinement leads to divergent numerical results for acoustic waves under a suddenly applied load. This is explained by large spurious high-frequency oscillations. For the quantification and the suppression of spurious oscillations, we have modified and applied a two-stage time-integration technique that includes the stage of basic computations and the filtering stage. This technique allows accurate convergent results at mesh refinement as well as significantly reduces the numerical anisotropy of solutions. We should mention that the approach suggested is very general and can be equally applied to any loading as well as for any space-discretization technique and any explicit or implicit time-integration method.

1. Introduction

In the paper we will consider the propagation of acoustic waves in an isotropic homogeneous medium described by the following scalar wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c_o^2 \nabla^2 u = f, \tag{1}$$

where u is the field variable, c_o is the wave velocity, f is the body force, and t is the time. The application of the space discretization to Eq. (1) leads to a system of ordinary differential equations in time

$$\boldsymbol{M}\,\ddot{\boldsymbol{U}} + \boldsymbol{K}\,\boldsymbol{U} = \boldsymbol{R}, \tag{2}$$

where $\boldsymbol{M}$ and $\boldsymbol{K}$ are the mass and stiffness matrices, respectively, $\boldsymbol{U}(t)$ is the vector of the nodal field variable, and $\boldsymbol{R}(t)$ is the vector of the nodal load. Due to the space discretization, the exact solution to Eq. (2) contains the numerical dispersion error; e.g., see [1–15] and others. The space discretization error can be decreased by the use of mesh refinement. However, this procedure significantly increases computational costs. Therefore, special techniques have been developed for the reduction in the numerical dispersion error which is also related to "the pollution effect" (e.g., see [16–18] and others for the study of the pollution error). One simple and effective technique for acoustic and elastic wave propagation problems is based on the calculation of the mass matrix $\boldsymbol{M}$ in Eq. (2) as a weighted average of the consistent and lumped mass matrices; see [5–8] and others. For the 1-D case and linear finite elements, this approach reduces the error in the wave velocity for harmonic waves from the second order to the fourth order of accuracy. However, for harmonic wave propagation in the 2-D and 3-D cases, these results are not valid (nevertheless, in the multi-dimensional case, the averaged mass matrix yields more accurate results compared with the standard mass matrix). We should also mention that the known publications on the techniques with the averaged mass matrix do not include the effect of finite time increments on the dispersion error and on the accuracy of the numerical

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results. As shown in the current paper, if we use the weighting coefficients for the averaged mass matrix that are independent of time increments (as in the known approaches) and if the time increments for explicit time-integration methods are close to the stability limit then there are no advantages in the use of the averaged mass matrix compared with the lumped mass matrix.

An interesting technique with implicit and explicit time-integration methods is suggested in [10] for acoustic waves. It is based on the modified integration rule for the calculation of the mass and stiffness matrices for linear finite elements. In contrast to the averaged mass matrix, the use of the modified integration rule increases the accuracy for the phase velocity from the second order to the fourth order in the general multi-dimensional case of acoustic waves. However, the effect of the size of time increments on the accuracy of numerical results has not been studied in [10]. The technique in [10] has not also treated spurious oscillations that may significantly destroy the accuracy of numerical results and lead to divergent results.

We should mention that the analysis of numerical dispersion estimates the numerical error for propagation of harmonic waves only. In the general case of loading (boundary conditions), the estimation of the accuracy of numerical techniques with reduced dispersion is difficult due to the presence of spurious high-frequency oscillations in numerical solutions; e.g., see [3,5].

In our previous papers [19–25], we have developed the accurate finite element techniques for elastodynamics problems. In this paper we have extended these techniques to acoustic waves. Because for wave propagation problems small time increments are necessary for accurate numerical results, explicit time-integration methods are more effective and require less computation time than implicit time-integration methods. The paper consists of a modification of the system of semidiscrete equations Eq. (2) that can be used with explicit time-integration methods and the finite elements with reduced dispersion (Section 2); the analytical study of numerical dispersion and the effect of time increments on the dispersion error for the modified integration rule, for the averaged mass matrix technique and for the standard approach with the lumped mass matrix that are used with the linear finite elements and the explicit time-integration method (Section 2); a short description and the modification of the two-stage time-integration technique with the filtering of spurious oscillations that is suggested in our previous papers [19–24] (Appendix); and 1-D and 2-D numerical examples showing the efficiency of the new two-stage time-integration technique for the simulation of acoustic waves under a suddenly applied load (Section 3).

2. Dispersion analysis

In this section, we will analyze the averaged mass matrix technique, the modified integration rule technique and the standard approach with the lumped mass matrix that are used with explicit time-integration methods. The first two techniques reduce the numerical dispersion error and the computation time compared with the standard finite element formulations for acoustic waves. In contrast to the study of the averaged mass matrix technique and the modified integration rule technique for the scalar wave equation considered in [5,10], here we will also analyze the effect of the size of time increments on the dispersion error. Similar to the paper [5], we will first modify Eq. (2) for the use of explicit time-integration methods. Let us rewrite Eq. (2) with the diagonal (lumped) mass matrix $\boldsymbol{D}$ as follows:

$$\boldsymbol{D}\,\dot{\boldsymbol{V}} + \boldsymbol{K}\,\boldsymbol{U} = \boldsymbol{R}, \tag{3}$$

where $\boldsymbol{V}$ is the vector of the nodal velocities (the time derivative of the nodal field variables). Relationships between the nodal field variables and their velocities can be written down as (similar to those in [5,10])

$$\boldsymbol{D}\,\dot{\boldsymbol{U}} = \boldsymbol{M}\,\boldsymbol{V} \quad \text{or} \quad \boldsymbol{D}\,\ddot{\boldsymbol{U}} = \boldsymbol{M}\,\dot{\boldsymbol{V}} \tag{4}$$

where $\boldsymbol{M}$ is the non-diagonal mass matrix calculated by the averaged mass matrix technique (see Eq. (8) below) or by the modified integration rule technique (see Eq. (9) below). Inserting Eq. (4) into Eq. (3) we will get

$$\boldsymbol{D}\,\ddot{\boldsymbol{U}} + \boldsymbol{M}\,\boldsymbol{D}^{-1}\,\boldsymbol{K}\,\boldsymbol{U} \;=\; \boldsymbol{M}\,\boldsymbol{D}^{-1}\,\boldsymbol{R}. \tag{5}$$

Eq. (5) differs from the standard finite element equations with the lumped mass matrix by the stiffness matrix and the load vector which are multiplied by the term $\boldsymbol{M}\,\boldsymbol{D}^{-1}$. For the time integration of Eq. (5) we will use the standard explicit central-difference method (the most popular explicit method); e.g., see [23,26,27]. Replacing the second time derivative in Eq. (5) by the corresponding finite difference approximation used in the central-difference method, we obtain

$$\boldsymbol{D}\,[\boldsymbol{U}(t+\Delta t) - 2\boldsymbol{U}(t) + \boldsymbol{U}(t-\Delta t)]/\Delta t^2 + \boldsymbol{M}\,\boldsymbol{D}^{-1}\,\boldsymbol{K}\,\boldsymbol{U}(t) = \boldsymbol{M}\,\boldsymbol{D}^{-1}\,\boldsymbol{R}(t), \tag{6}$$

where Δt is the time increment. We will use Eq. (6) for the analysis of the numerical dispersion of the finite element formulations with the averaged mass matrix and modified integration rule techniques. One of the main difficulties in the analytical study of numerical dispersion is the complicated non-linear structure of the equation for the numerical phase velocity. Therefore, in order to simplify the dispersion analysis, we will study the residual of this equation instead of the direct calculation of the numerical phase velocity.

Propagation of harmonic plane waves in an infinite medium with

$$u(\boldsymbol{x}, t) = \bar{u} exp(i\,k(\boldsymbol{n}\cdot\boldsymbol{x} \pm c_o\,t)) \tag{7}$$

is used for the dispersion analysis. Here, $\bar{u}$ is the amplitude of the acoustic waves, $\boldsymbol{x}$ is the position vector, k is the wave number, $\boldsymbol{n}$ is the unit normal to the wave front; $\boldsymbol{k} = k\,\boldsymbol{n}$ is the wave vector, c_o is the phase velocity, t is the time, and $i = \sqrt{-1}$. It can be easily checked that Eq. (7) meets Eq. (1) with $f = 0$; i.e., harmonic plane waves propagate with the constant wave velocity c_o and an isotropic infinite homogeneous medium is non-dispersive.

Below we study the numerical dispersion of linear quadrilateral finite elements on uniform meshes in the 2-D case. Because acoustic waves propagate in all directions with the same velocity, then finite elements with the same dimensions along the coordinate axes can be recommended for wave propagation problems in the multi-dimensional case (e.g., see [1,6,10] and others). They are also used in the current paper. In order to decrease the dispersion of finite element results, we consider the following two possibilities for the calculation of the mass and stiffness matrices: the mass matrix $\boldsymbol{M}$ is calculated as a weighted average of the consistent $\boldsymbol{M}^{cons}$ and lumped $\boldsymbol{D}$ mass matrices with the weighting factor γ (similar to that used in [5,6,8])

$$\boldsymbol{M}(\gamma) = \boldsymbol{D}\gamma + \boldsymbol{M}^{cons}(1-\gamma) \tag{8}$$