ARTICLE IN PRESS

Computer Physics Communications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computer Physics Communications

journal homepage: www.elsevier.com/locate/cpc

Nonextensive lattice gauge theories: Algorithms and methods

Rafael B. Frigori *

Universidade Tecnológica Federal do Paraná (UTFPR), Rua Cristo rei 19, CEP 85902-490, Toledo (PR), Brazil

ARTICLE INFO

Article history:
Received 20 September 2013
Received in revised form
21 April 2014
Accepted 25 April 2014
Available online xxxx

Keywords:
Dynamic critical phenomena
Lattice gauge theory
Algorithms

ABSTRACT

High-energy phenomena presenting strong dynamical correlations, long-range interactions and microscopic memory effects are well described by nonextensive versions of the canonical Boltzmann–Gibbs statistical mechanics. After a brief theoretical review, we introduce a class of generalized heat-bath algorithms that enable Monte Carlo lattice simulations of gauge fields on the nonextensive statistical ensemble of Tsallis. The algorithmic performance is evaluated as a function of the Tsallis parameter q in equilibrium and nonequilibrium setups. Then, we revisit short-time dynamic techniques, which in contrast to usual simulations in equilibrium present negligible finite-size effects and no critical slowing down. As an application, we investigate the short-time critical behaviour of the nonextensive hot Yang–Mills theory at q-values obtained from heavy-ion collision experiments. Our results imply that, when the equivalence of statistical ensembles is obeyed, the long-standing universality arguments relating gauge theories and spin systems hold also for the nonextensive framework.

1. Introduction

There is increasing evidence that generalizations of the canonical thermostatistics of Boltzmann–Gibbs (BG) are useful to describe important phenomenological aspects of relativistic hadronic collisions [1,2]. Traditionally, a QCD inspired formula à la Hagedorn [3] is employed to fit the cross sections (σ) of hadrons as a function of their transverse momenta (p_T)

$$E\frac{d^3\sigma}{d^3p} = C \cdot \left(1 + \frac{p_T}{p_0}\right)^{-\alpha} \rightarrow \begin{cases} p_T \rightarrow 0 \Longrightarrow \exp\left(-\alpha p_T/p_0\right) \\ p_T \rightarrow \infty \Longrightarrow \left(\dfrac{p_T}{p_0}\right)^{\alpha} \end{cases} \quad (1)$$

with parameters C, p_0 and α. Where the mean transverse momentum of the system $\langle p_T \rangle$ is related to its hadronization temperature T in equilibrium.

However, such temperature shall naturally fluctuate among events [4], in a clear far-from-equilibrium scenario. Therefore, the usual BG picture shall be generalized to naturally accommodate such fluctuations, this is done by considering a Tsallis [5,6] distribution

$$E\frac{d^3\sigma}{d^3p} = C_q \cdot \left[1 - (1-q)\,\frac{p_T}{T}\right]^{\frac{1}{1-q}}. \quad (2)$$

Here $\alpha = \frac{1}{q-1}$, $p_0 = \frac{T}{q-1}$ and C_q is a normalization, whereas the nonextensive parameter q is related [7] to the variance of T by

$$q = 1 + \frac{Var\,(T)}{\langle T\rangle^2}. \quad (3)$$

Along the last years variations of the approach in Eq. (2) have been verified by different collaborations, as ALICE [8], ATLAS [9] and CMS [10] at LHC and PHENIX [11] and STAR [12] at RHIC, which has fit experimental data by power-like (Levy) distributions using Tsallis formulae [13,14].

While Eqs. (1) and (2) may seem similar from the mathematical point, their underlying physics is quite distinct. The nonextensive expression admits a complex (steady state) thermal equilibrium for any p_T, which can be described by just two parameters T and q. Thus, it is a unifying statistical mechanics approach that does not rely on any particular model, or theoretical regime of a more fundamental theory (i.e. perturbative vs nonperturbative QCD) [2], to be derived from.

Still, this conceptual difference plays a central role when modelling heavy-ion collisions through a hydrodynamical approach with evolution [15]. There, the transverse momentum distributions of multiple particles species is usually described by a Boltzmann–Gibbs Blast-Wave (BGBW) model, see [16] (and references therein). This brings most of physical insights about the behaviour of the fireball. Nevertheless, such an equilibrium description is believed to break at high p_T, when nonequilibrium effects and hard processes will exhibit power-law tail [7]. So, generalizations of BGBW incorporating principles of Tsallis thermostatistics [16]

* Tel.: +55 45 33796853.
E-mail addresses: frigori@utfpr.edu.br, frigori@yahoo.com.

http://dx.doi.org/10.1016/j.cpc.2014.04.016
0010-4655/

Please cite this article in press as: R.B. Frigori, Nonextensive lattice gauge theories: Algorithms and methods, Computer Physics Communications (2014), http://dx.doi.org/10.1016/j.cpc.2014.04.016

are necessary (TBW), and in fact they have shown to be powerful enough to describe experimental data [1,11,17].

While nonextensive extensions of well-known phenomenological models is an attractive area, with potential implications [18] – see also [19] for q-Walecka and [20] for q-NJL – their derivation from first principles is not fully understood yet [21,22]. Therefore, it is of high theoretical interest to generalize first-principle nonperturbative methods, as the lattice formalism of non-abelian gauge theories [23], to the Tsallis ensemble. Moreover, from a pure computational perspective, lattice simulations may benefit from Tsallis weight, hence it enhances the tunnelling rate among metastable states during phase transitions [24].

In this context, following the generalized master equation approach of [25], we introduce a generalized hybrid heat-bath algorithm to enable Monte Carlo simulations of lattice Yang–Mills (YM) theory in the Tsallis ensemble. In addition, this algorithm can be also easily adapted to other $SU(N)$ gauge theories. Thus, we perform a rigorous analysis of algorithmic performance in 2d lattices, where the $SU(2)$ theory is exactly solvable. This solution helps on evaluating integrated correlation-times of critical plaquettes, as a function of the lattice-side and q, while investigating a region of *constant physics* [26]. In agreement with the previous studies [24,27] we observe that setups with $q > 1$ induce significant improvements on computational efficiency when compared to the canonical case ($q = 1$).

On the other hand, when considering finite-temperature simulations on equilibrium, universality has been a cornerstone principle to understand the thermodynamics of gauge theories in the canonical ensemble. For instance, even dynamical aspects of such theories, as their screening-mass spectra [28], were predicted from condensed-matter analogous. Also, based on arguments of symmetry, QCD with two dynamic quarks undergoes a phase transition with universal critical scaling in the class of the 3d $O(4)$ continuous-spin model[1] [29]. Furthermore, there is the long-standing argument by Svetitsky and Yaffe [30] relating critical quenched $SU(N)$ gauge fields in $d+1$ dimensions to Z_N spin systems in d-dim.

Those simulations are challenging, not only because finite-size (FS) effects – and their necessary scaling extrapolations – have to be kept under control, but also because the well-known critical slowing down (CSD) effect [31]. This implies exponentially diverging correlation times of observables, and so their statistical errors. A way to alleviate that computational burden comes from short-time dynamical simulations, for a review see [32,33]. This technique allows for extracting the critical behaviour, summarized in a set of dynamic and static exponents, of spin-systems or gauge fields without appreciable FS or CSD effects. This feature is rooted on the findings [34] that even during a short-time transient regime, before (Monte Carlo) equilibration happens, the Hamiltonian dynamics already exhibits universal scaling.

Considering that few is known about the aforementioned critical properties of nonextensive gauge theories, we investigate through short-time simulations the finite-temperature 3d $SU(2)$ lattice Y.M. theory $\left(YM_2^{3d}\right)$ in the Tsallis ensemble. Despite of being considerably simpler than unquenched QCD, the YM_2^{3d} theory is nontrivial. Actually, it has been shown to be a good theoretical model for understanding fundamental properties of confinement. Concerning gluonic propagators, no relevant discrepancies to QCD were found at gauge groups [35] or dimensionality [36] levels. In addition, around criticality YM_2^{3d} is related to the bidimensional Ising model, an exactly solvable system, which turns that gauge theory more auspicious for high-precision comparative studies.

[1] Incidentally, this spin system can be simulated using a heat-bath algorithm shared by lattice Y.M. theory [26].

As a matter of fact, we have focused our simulations on values of the nonextensive parameter $q \approx 1$ – as a perturbation around BG thermodynamics – and $q = 1.10$, a value favoured by experimental data fits [1,2,13]. We have observed that for $q \approx 1$ small deviations from usual BG behaviour are seen. While in the $q > 1$ regime the temperature of the phase-transition is monotonically increased with q (i.e. $T^{crit.}_{q>1} > T^{crit.}_{q=1}$), as theoretically expected [2,7]. Besides that, by performing a Binder cumulant analysis (in equilibrium) [37] we confirm that our results are not afflicted by any FS effect. More interestingly, not only the static and dynamic exponents of the nonextensive theory, but also its universal cumulant values, can be explained by (and generalizes) universality arguments [30].

The article is organized as follows: in Section 2 the nonextensive thermostatistics of Tsallis is outlined. Its connections with the usual Boltzmann–Gibbs statistics are discussed in the sense of superstatistics, and finally, applications to gauge theories are provided. Section 3 reviews short-time dynamic simulation techniques for gauge theories. It gives an outlook on how to overcome the critical slowing down phenomena, while evaluating static and dynamic exponents. Our generalized algorithmic proposal is presented in Section 4, after briefly reviewing the theory of Markov processes and (generalized) detailed balance. The necessary modifications to usual heath-bath updating engines [26] is theoretically motivated and implemented. In Section 5, numerical results on algorithmic performance are analyzed for the 2d $SU(2)$ gauge theory and, the nonextensive relaxation dynamics for finite-temperature 3d $SU(2)$ theory is studied. Main conclusions and prospective research directions are the focus of Section 6.

2. Nonextensive thermostatistics of lattice gauge theories

The lattice gauge theory formalism allows for *ab initio* thermodynamic analysis of quantum fields at finite-temperature nonperturbative regimes [23]. Most times it is performed in the quenched approximation, where quark-loop effects are neglected. Within this approach the deconfinement phase transition of $SU(N)$ theories can be related by universality arguments to the magnetic transition of Z_N spin models [30].

A realization for pure gauge $SU(N)$ theories in d-dimensional lattices is given [23] by the Wilson action

$$S_W[U] \equiv \beta \sum_x \sum_{\mu,\nu=1}^{d} \left\{1 - \frac{1}{N} Re(TrP_{\mu\nu})\right\}, \tag{4}$$

where gauge links $U_\mu(x) \in SU(N)$ are combined to build a gauge-invariant plaquette

$$P_{\mu\nu} \equiv U_\mu(x)\, U_\nu\left(x + \hat{\mu}a\right) U_\mu^{-1}\left(x + \hat{\nu}a\right) U_\nu^{-1}(x). \tag{5}$$

The lattice-coupling $\beta = 2N/g_s^2 a^{4-d}$ is set in terms of the gauge-field coupling g_s and the physical lattice spacing a.

In the canonical ensemble the temperature of equilibrium is identified with the inverse length of the temporal direction (i.e. $T^{-1} = a \cdot L_t$) of an asymmetric lattice, whose volume is $V = a^d \cdot L_s^{d-1} \cdot L_t$ [23]. Thus, thermal expectation values of any gauge-invariant operator $\mathcal{O}$ may be computed by

$$\langle \mathcal{O} \rangle_{BG} = \frac{\sum_U \mathcal{O}(U)\, e^{-S_W(U)}}{\sum_U e^{-S_W(U)}}. \tag{6}$$

Among such observables is the (spatially averaged) Polyakov loop $\bar{W} \equiv \langle W(x,y,z) \rangle = \left\langle Tr \prod_{n=1}^{n=L_t} U_t(x,y,z,an) \right\rangle$, the order parameter of deconfinement phase transition.