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A composite numerical scheme for the numerical simulation of coupled Burgers' equation

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ABSTRACT

In this work, a composite numerical scheme based on finite difference and Haar wavelets is proposed to solve time dependent coupled Burgers' equation with appropriate initial and boundary conditions. Time derivative is discretized by forward difference and then quasilinearization technique is used to linearize the coupled Burgers' equation. Space derivatives discretization with Haar wavelets leads to a system of linear equations and is solved using Matlab7.0. Convergence analysis of proposed scheme exhibits that the error bound is inversely proportional to the resolution level of the Haar wavelet. Finally, the adaptability of proposed scheme is demonstrated by numerical experiments and shows that the present composite scheme offers better accuracy in comparison with other existing numerical methods.

1. Introduction

Nonlinear phenomena play a crucial role in various fields of science and engineering; therefore analytic or numerical solutions are essential for corresponding nonlinear equations. Apart from a limited number of these problems, most of them do not have a precise analytical solution, so the solution of these nonlinear equations using numerical methods is fundamentally important. Here we analyzed one of the most important nonlinear phenomena modeled as a coupled Burgers' equation. Due to the nonlinear convection term and viscosity term, the coupled Burgers' equation can be studied as simple case of the Navier–Stokes equation.

The coupled nonlinear viscous Burgers' equation is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \eta\, u \frac{\partial u}{\partial x} - \alpha \frac{\partial}{\partial x}(u\, v), \quad x \in [a,\ b],\ t \in [0, T] \tag{1}$$

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - \xi\, v \frac{\partial v}{\partial x} - \beta \frac{\partial}{\partial x}(u\, v), \quad x \in [a,\ b],\ t \in [0, T] \tag{2}$$

with initial conditions

$$u(x,\ 0) = g_1(x), \qquad v(x,\ 0) = g_2(x), \quad x \in [a, b] \tag{3}$$

and boundary conditions

$$\begin{aligned} u(a,t) &= f_1(t), \qquad u(b,t) = f_2(t), \\ v(a,t) &= f_3(t), \qquad v(b,t) = f_4(t), \end{aligned} \tag{4}$$

where $\eta,\ \xi$ are real constants and α, β are arbitrary constants depending upon the system parameters such as the Péclet number. $u(x,t)$ and $v(x,t)$ are the velocity components to be determined. $\partial u/\partial t$ is unsteady term; $u(\partial u/\partial x)$ is the nonlinear convection term and $\partial^2 u/\partial x^2$ is the diffusion term. The functions $g_1(x),\ g_2(x),\ f_1(t),\ f_2(t),\ f_3(t),\ f_4(t)$ are sufficient smooth functions.

It is well known that the nonlinear coupled Burgers' equation does not possess precise analytic solutions. Scientists and engineers are interested in studying the properties of the Burgers' equation using various numerical techniques due to its wide range of applicability in respective fields of science and engineering. The numerical solutions of Eqs. (1) and (2) have great importance due to their applications in approximate theory of flow through a shock wave traveling in a viscous fluid and in the model of polydispersive sedimentation [1]. Recently, contributed works associated with time dependent coupled viscous Burgers' equations have been published that analyze theoretical and numerical aspects. Solutions of nonlinear evolution equations have been studied using analytical methods, such as Laplace transform method, Tanh method [2], Fourier transform method [3], Hirota's bilinear transform method [4,5], Sine–cosine method, Exp-function method, etc.

Numerical Approximation of nonlinear coupled Burgers' equation is a challenging task due to the presence of nonlinear terms

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and viscosity parameters. So far some numerical techniques such as fully implicit finite difference scheme [6], variational iteration method [7], adomian decomposition method [8], differential transformation method [9], cubic B-spline collocation scheme [10] and differential quadrature methods [11–15], Chebyshev spectral collocation method [16] and analytic method [17] have been developed for the numerical solutions of Burgers' and coupled Burgers' equations.

Due to the non-availability of an analytical solution in the literature, the numerical solution of the coupled Burgers' equation is very important for researchers in the present age. Rashid et al. [3] provide an approximate solution for coupled Burgers' equation with a set of initial and periodic boundary conditions through the Fourier pseudo-spectral (FPS) method. A fully implicit finite-difference method has been implemented in [6] for the solution of a nonlinear coupling and compared the numerical results of the coupled Burgers' equation for different time levels with different numbers of partitions.

In [10] the authors constructed a collocation method using cubic B-spline functions for the numerical simulation of coupled Burgers' equation. The time derivative terms are discretized by the usual finite difference Crank–Nicolson scheme and space derivative terms are discretized using cubic B-spline method. The result obtained through cubic B-spline with finite difference demonstrated that the accuracy of the solution reduces as time increases due to the time truncation errors of the time derivative term.

Mittal et al. [11] analyzed a coupled Burgers' equation through differential quadrature method (DQM). Using DQM coupled Burgers' equation transforms into system of ordinary differential equations. The obtained system of ODEs is then solved by the Runge–Kutta method. It is also demonstrated that DQM can easily be applied to solve these kinds of nonlinear partial differential equations.

Khater et al. [16] employed Chebyshev spectral collocation method for the approximate solution of nonlinear evolution equations. Using Chebyshev spectral collocation method, partial differential equation reduces into system of ordinary differential equations that are solved by Runge–Kutta method of order four. The results obtained using Chebyshev collocation method have been compared with the exact solution, finite difference method and Galerkin quadratic B-spline finite element method.

The compactly supported orthonormal Haar wavelets are groups of square waves with magnitudes 1, −1 in some intervals and zero elsewhere, but the technical disadvantage of the Haar wavelet is that it is not continuous and therefore it is not differentiable. There are two possibilities for coming out of this situation. One way is to regularize the Haar wavelets with interpolating splines and the other way is integral methods [18–21]. We introduce the Haar wavelet integral method to solve a coupled Burgers' equation because it has several advantageous feature such as high accuracy, fast transformation, small computation cost, etc.

In the present paper, we extended the scope of applicability of the method [19] to a coupled Burgers' equation by combining finite difference and Haar wavelets. In this composite scheme, first of all, we discretized time derivative terms by forward difference and linearized the nonlinear terms using a quasilinearization technique [22] to reduce the original equation into a system of ordinary differential equations. Then we applied the Haar wavelet method which leads to a system of algebraic equations. To solve the system of algebraic equations we used the back slash operator in Matlab. The proposed method not only widens the area of applicability but also reduces the computational time and improves the accuracy of the algorithm. The error analysis of this composite scheme shows that the error bound is inversely proportional to the resolution level of Haar wavelets. Finally, the accuracy of the proposed scheme is demonstrated on two test problems and the results are compared with other existing methods [3,6,10,11,16] and it is shown that the present numerical scheme gives better solutions.

In order to elucidate our arguments in a synchronized manner we have summed up the paper as follows. Section 2 presents the brief description of Haar wavelets and its integrals. In Section 3, the discretization of time derivative terms with forward difference and quasilinearization technique for nonlinear terms have been given. Section 4 describes the Haar wavelets method for spatial derivatives. We introduce the Error analysis for the composite scheme in Section 5. In Section 6, numerical examples are included to demonstrate the physical behaviors of coupled Burgers' equation via 3-dimensional graphs and contour plots. Numerical outcomes are given in Section 7 as a conclusion.

2. Preliminaries of Haar wavelets

Recently, the Haar wavelet is being used as a mathematical tool for solving many problems arising in science and engineering. There are different definitions of Haar function and various generalizations have been described in the literature [18,23,19–21,24,25]. For $x \in [0,\ 1)$ the family of Haar wavelets is defined as

$$h_i(x) = \begin{cases} 1 & x \in [\xi_1,\ \xi_2], \\ -1 & x \in [\xi_2,\ \xi_3], \\ 0 & otherwise \end{cases} \tag{5}$$

where

$$\xi_1 = \frac{k}{m}, \qquad \xi_2 = \frac{k+0.5}{m}, \qquad \xi_3 = \frac{k+1}{m} \tag{6}$$

and level of wavelet $m = 2^l$, $(l = 0, 1, \ldots, J)$, translation parameter $k = 0,\ 1, \ldots, m-1$. J indicates the maximal level of resolution. The index i in the Eq. (5) is given by $i = m + k + 1$. For minimal values $m = 1,\ k = 0$, we have $i = 2$ and the maximum value of i is $i = 2M = 2^{J+1}$. For $i = 1$, the function $h_1(x)$ is a scaling function for the family of the Haar wavelets as

$$h_1(x) = \begin{cases} 1 & x \in [0,\ 1) \\ 0 & otherwise. \end{cases} \tag{7}$$

To solve the nth order partial differential equation, we need the following integrals

$$p_{n,i}(x) = \int_0^x \int_0^{t_{n-1}} \cdots \int_0^{t_1} h_i(t)\ dt\, dt_1 \ldots dt_{n-1}, \quad i = 1, 2, \ldots, 2M \tag{8}$$

using Eqs. (5) and (8) we have

$$p_{n,i}(x) = \begin{cases} 0, & x < \xi_1 \\ \dfrac{(x-\xi_1)^n}{n!}, & x \in [\xi_1,\ \xi_2) \\ \dfrac{(x-\xi_1)^n}{n!} - 2\dfrac{(x-\xi_2)^n}{n!}, & x \in [\xi_2,\ \xi_3) \\ \dfrac{(x-\xi_1)^n}{n!} - 2\dfrac{(x-\xi_2)^n}{n!} + \dfrac{(x-\xi_3)^n}{n!}, & x > \xi_3. \end{cases} \tag{9}$$

This formula holds for $i > 1$.

3. Semi-discretization of the coupled Burgers' equation

To apply the proposed composite scheme, we discretize the time derivatives using forward finite difference to Eqs. (1) and (2), we have

$$u_{j+1} - u_j = \Delta t\,(u_{xx})_{j+1} - \eta\,\Delta t(u\,u_x)_{j+1} - \alpha\,\Delta t\,(u_x\,v + u\,v_x)_{j+1} \tag{10}$$

$$v_{j+1} - v_j = \Delta t\,(v_{xx})_{j+1} - \xi\,\Delta t(v\,v_x)_{j+1} - \beta\,\Delta t\,(u_x\,v + u\,v_x)_{j+1} \tag{11}$$