Engineering Analysis with Boundary Elements 36 (2012) 248–254

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Analytical integration of the moments in the diagonal form fast multipole boundary element method for 3-D acoustic wave problems

Haijun Wu[a], Yijun Liu[b], Weikang Jiang[a,*]

[a] State Key Laboratory of Mechanical System and Vibration, Shanghai Jiao Tong University, Shanghai 200240, China
[b] Mechanical Engineering, University of Cincinnati, Cincinnati, OH 45221-0072, USA

ARTICLE INFO

Article history:
Received 26 February 2011
Accepted 28 July 2011
Available online 8 October 2011

Keywords:
3-D acoustic wave problems
Helmholtz equation
Boundary element method
Fast multipole method
Diagonal form
Analytical integration

ABSTRACT

A diagonal form fast multipole boundary element method (BEM) is presented in this paper for solving 3-D acoustic wave problems based on the Burton–Miller boundary integral equation (BIE) formulation. Analytical expressions of the moments in the diagonal fast multipole BEM are derived for constant elements, which are shown to be more accurate, stable and efficient than those using direct numerical integration. Numerical examples show that using the analytical moments can reduce the CPU time by a lot as compared with that using the direct numerical integration. The percentage of CPU time reduction largely depends on the proportion of the time used for moments calculation to the overall solution time. Several examples are studied to investigate the effectiveness and efficiency of the developed diagonal fast multipole BEM as compared with earlier p^3 fast multipole method BEM, including a scattering problem of a dolphin modeled with 404,422 boundary elements and a radiation problem of a train wheel track modeled with 257,972 elements. These realistic, large-scale BEM models clearly demonstrate the effectiveness, efficiency and potential of the developed diagonal form fast multipole BEM for solving large-scale acoustic wave problems.

1. Introduction

The boundary element method (BEM) based on the boundary integral equation (BIE) formulation can be used to analyze acoustic wave problems effectively, such as in noise prediction for automobiles [1], high speed trains [2], airplanes [3], and underwater structures [4]. Several of the earlier work laid the foundation for applying the BIE/BEM to solve acoustic problems [5–10]. Especially, the work by Burton and Miller in Ref. [7] has been regarded as a classical one, which provides an elegant way to overcome the so called fictitious eigenfrequency difficulties existing in the conventional BIE for exterior acoustic wave problems [11,12].

In the last decade, the focus of the research has been on developing fast solution methods for efficiently solving large-scale BEM models for acoustic problems. The fast multipole method (FMM) is one of the most promising fast solution methods for the BEM. FMM was first pioneered by Rokhlin [13] and further developed by Greengard and Rokhlin [14] for fast simulation of large particle fields in physics. The FMM can improve the matrix-vector multiplication dramatically from $O(N^2)$ to $O(N)$ or $O(N \log N)$ with N being the number of degrees of freedom. Later on, a diagonal form FMM for Helmholtz problems was proposed by Rokhlin [15] as well. Since then, many research works have been published to improve and extend the applicability of the FMM for Helmholtz equations. Epton and Dembart [16] presented a concise summary of multipole translations for 3-D Helmholtz equations. Rahola [17] gave an error analysis of the FMM by considering both truncation error of the kernel expansion and the errors from the use of numerical integration in diagonal translation theorem. Darve [18] provided a rigorous mathematical approach on the estimation of the truncation error. Besides the above error considerations, Koc et al. [19] also analyzed the interpolation error in multilevel FMM. To accelerate the low frequency FMM, Greengard et al. [20] used the combination of evanescent and propagate mode to reduce the computation cost. Darve and Have [21] proposed a stable plane wave expansion, which uses the singular-value decomposition method to represent the evanescent kernel for the low frequency FMM. Gumerov and Duraiswami [22,23] extended the recurrence relations reported in Chew's paper [24] to develop a general recursive method for obtaining the translation matrices, the resulting approach is generally termed as p^3 FMM for solving the Helmholtz equation (with p being the order of the expansion). Adaptive algorithms for the FMM were also developed to speed up the solutions for 3-D full- and half-space acoustic problems [25–27]. The fast multipole BEM for solving structural-acoustic interaction problems was developed by

* Corresponding author. Tel.: +86 2134206332 824; fax: +86 21 34205783.
E-mail address: wkjiang@sjtu.edu.cn (W. Jiang).

0955-7997/$ - see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.enganabound.2011.08.004

Gaul and Fisher [28,29]. Hybrid FMMs were developed recently by Cheng et al. [30] and Gumerov and Duraiswami [31], which are stable for a wide range of frequencies. The former switches to different representations at low and high frequencies, while the latter is based on a rotation – coaxial translation – back rotation scheme. More information about the fast multipole BEM in general can be found in a review article [32], a tutorial [33], and the first textbook [34].

A new diagonal fast multipole BEM for solving 3-D Helmholtz equation with analytical integration of the moments is presented in this paper. The BEM is based on the Burton–Miller's BIE formulation [7], which has no fictitious eigenfrequency difficulties in solving exterior acoustic problems. The implementation of the diagonal FMM is also based on the adaptive fast multipole BEM given in Ref. [25] for 3-D full-space acoustic problems, in Ref. [26] for 3-D half-space acoustic problems, and in Ref. [27] for a new definition of the interaction list. The FMM used in Refs. [25–27] is valid for all frequencies, but is less efficient than the diagonal form FMM at high frequencies (e.g., with the nondimensional wavenumber ka above 300), since the translation complexity is at best $O(p^3)$ in Refs. [25–27] (with p being the expansion order at each tree level). The developed diagonal fast multipole BEM with the analytical integration of the moments is a significant improvement of the above mentioned fast multipole BEM, which can fill the gap in the analysis of high-frequency acoustic problems.

The rest of the paper is organized as follows. First the BIE formulation is reviewed in Section 2. The diagonal form FMM is presented in Section 3. Then, the fast multipole BEM algorithm is described in Section 4. The analytical moment formulation for the diagonal FMM is presented in Section 5. In Section 6, several numerical examples are given to demonstrate the capability of the proposed diagonal form FMM in modeling large-scale acoustic problems. Section 7 concludes this paper.

2. Boundary integral equations

The governing equation in the frequency domain of time-harmonic acoustic waves in a homogeneous isotropic acoustic medium E is described by the following Helmholtz equation:

$$\nabla^2 \varphi(\mathbf{x}) + k^2 \varphi(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in E, \tag{1}$$

where $\varphi(\mathbf{x})$ is the sound pressure at point $\mathbf{x}$, k is the wave number defined by $k=\omega/c$, with ω being the angular frequency and c the sound speed in medium E. Using Green's second identity, the solution of Eq. (1) can be expressed by an integral representation:

$$\varphi(\mathbf{x}) = \int_S \left[G(\mathbf{x},\mathbf{y}) q(\mathbf{y}) - \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})} \varphi(\mathbf{y}) \right] dS(\mathbf{y}) + \varphi^I(\mathbf{x}), \quad \forall \mathrm{x} \in E, \tag{2}$$

where $\mathbf{x}$ is the source point and $\mathbf{y}$ is the field point on boundary S, $q(\mathbf{y})$ is defined as $q(\mathbf{y})=\partial\varphi(\mathbf{y})/\partial n(\mathbf{y})$ where the unit normal vector $n(\mathbf{y})$ on boundary S is defined to point outwards from E. Incident wave $\varphi^I(\mathbf{x})$ will not be presented for radiation problems. In this paper, the time convention adopted is using the factor $e^{-i\omega t}$, correspondingly, the free-space Green's function G for 3-D problems is given by

$$G(\mathbf{x},\mathbf{y}) = \frac{e^{ikr}}{4\pi r} \text{ with } r = |\mathbf{x}-\mathbf{y}|. \tag{3}$$

Letting point $\mathbf{x}$ approach the boundary leads to the following conventional boundary integral equation (CBIE):

$$c(\mathbf{x})\varphi(\mathbf{x}) = \int_S \left[G(\mathbf{x},\mathbf{y}) q(\mathbf{y}) - \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})} \varphi(\mathbf{y}) \right] dS(\mathbf{y}) + \varphi^I(\mathbf{x}), \quad \forall \mathbf{x} \in S, \tag{4}$$

where constant $c(\mathbf{x})=1/2$ if S is smooth around point $\mathbf{x}$. There is a defect with Eq. (4) concerning the non-uniqueness of the solution of an exterior acoustic problem at the eigenfrequency associated with the corresponding interior problem. To deal with the non-uniqueness difficulties, Burton and Miller [7] proposed a method by combining the CBIE and the normal derivative of the CBIE. Taking the derivative of integral representation Eq. (2) with respect to the normal at the field point $\mathbf{x}$ and also letting point $\mathbf{x}$ approach the boundary lead to the following hypersingular boundary integral equation (HBIE):

$$c(\mathbf{x})q(\mathbf{x}) = \int_S \left[\frac{\partial G(x,y)}{\partial n(x)} q(\mathbf{y}) - \frac{\partial^2 G(x,y)}{\partial n(y) \partial n(x)} \varphi(\mathbf{y}) \right] dS(\mathbf{y}) + q^I(\mathbf{x}), \quad \forall \mathbf{x} \in E, \tag{5}$$

where $q^I(\mathbf{x}) = \partial\varphi^I(\mathbf{x})/\partial n(\mathbf{x})$. For an exterior problem, Eqs. (4) and (5) have a different set of fictitious frequencies at which unique solutions for the exterior problem cannot be obtained. However, a linear combination of Eqs. (4) and (5) will always have unique solutions [7]. That is, the following linear combination of Eqs. (4) and (5) (CHBIE) yields unique solutions at all frequencies:

$$\begin{aligned} &\beta \int_S \frac{\partial^2 G(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{x}) \partial n(\mathbf{y})} \varphi(\mathbf{y}) dS(\mathbf{y}) + \int_S \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})} \varphi(\mathbf{y}) dS(\mathbf{y}) + c(\mathbf{x})\varphi(\mathbf{x}) - \varphi^I(\mathbf{x}) \\ &\quad = \beta \left[q^I(\mathbf{x}) - c(\mathbf{x}) q(\mathbf{x}) + \int_S \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{x})} q(\mathbf{y}) dS(\mathbf{y}) \right] + \int_S G(\mathbf{x},\mathbf{y}) q(\mathbf{y}) dS(\mathbf{y}), \end{aligned} \tag{6}$$

where β is a coupling constant that must be a complex number and can be chosen, for example, as i/k. This CHBIE formulation is referred to as the Burton–Miller formulation.The acoustic problem considered in this paper is to solve Eq. (6) with the fast multipole BEM under given boundary conditions.

3. Diagonal form fast multipole method

The FMM is employed to solve the Burton–Miller BIE, or CHBIE (6), for which iterative solver GMRES will be used. Two earlier versions of the FMM are available in the literature. One is based on a multipole expansion of the kernel, named low frequency method, and another based on a plane wave expansion of the kernel, referred as the diagonal form method. Both of them have their drawbacks. It is costly and sometimes not applicable to perform low frequency fast multipole BEM in the high frequency regime. On the other hand, due to the divergence of the translations when the size of the clusters becomes very small compared with the wavelength and round-off errors of the translations, the diagonal form is unstable when it is used in the low frequency range. Despite their limitations, those methods have been proved to be very successful in their suitable frequency ranges.

The diagonal form FMM is based on a plane wave expansion of the kernel, which can be described by the following expansion [17]:

$$G(\mathbf{x},\mathbf{y}) \approx \sum_{n=0}^{N_l} \frac{ik}{8\pi} \frac{\omega_n}{2N_l+1} \sum_{m=0}^{2N_l} I_n^m(k,\mathbf{x},\mathbf{x}_c) T_n^m(k,\mathbf{x}_c,\mathbf{y}_c) O_n^m(k,\mathbf{y}_c,\mathbf{y}), \tag{7}$$

for $|\mathbf{x}-\mathbf{x}_c| < |\mathbf{y}-\mathbf{x}_c|$ and $|\mathbf{y}-\mathbf{y}_c| < |\mathbf{x}-\mathbf{y}_c|$, where $\mathbf{x}_c$ is an expansion point near $\mathbf{x}$ and $\mathbf{y}_c$ is that near $\mathbf{y}$, N_l is the truncation number of the multipole expansion. The inner, translation and outer functions in Eq. (7) are defined by

$$I_n^m(k,\mathbf{x},\mathbf{x}_c) = e^{ik(\mathbf{x}-\mathbf{x}_c)\cdot\hat{\mathbf{s}}_{nm}}, \tag{8}$$

$$T_n^m(k,\mathbf{x}_c,\mathbf{y}_c) = \sum_{l=0}^{N_l} i^l (2l+1) h_l^{(1)}(ku) P_l(\hat{\mathbf{u}}\cdot\hat{\mathbf{s}}_{nm}), \tag{9}$$

$$O_n^m(k,\mathbf{y}_c,\mathbf{y}) = e^{ik(\mathbf{y}_c-\mathbf{y})\cdot\hat{\mathbf{s}}_{nm}}, \tag{10}$$

respectively, where $u=|\mathbf{x}_c-\mathbf{y}_c|$ and $\hat{\mathbf{u}} = (\mathbf{x}_c-\mathbf{y}_c)/u$, P_l is lth order Legendre function, $\hat{\mathbf{s}}_{nm} = (\sin\theta_n \cos\varphi_m, \sin\theta_n \sin\varphi_m, \cos\theta_n)$ in which