Engineering Analysis with Boundary Elements 36 (2012) 255–266

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

A posteriori error estimates for the Johnson–Nédélec FEM–BEM coupling

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ARTICLE INFO

Article history:
Received 25 February 2011
Accepted 27 July 2011
Available online 8 October 2011

Keywords:
Finite element-boundary element coupling
Local mesh-refinement
Adaptive algorithm

ABSTRACT

Only very recently, Sayas [The validity of Johnson–Nédélec's BEM-FEM coupling on polygonal interfaces. SIAM J Numer Anal 2009;47:3451–63] proved that the Johnson–Nédélec one-equation approach from [On the coupling of boundary integral and finite element methods. Math Comput 1980;35:1063–79] provides a stable coupling of finite element method (FEM) and boundary element method (BEM). In our work, we now adapt the analytical results for different a posteriori error estimates developed for the symmetric FEM–BEM coupling to the Johnson–Nédélec coupling. More precisely, we analyze the weighted-residual error estimator, the two-level error estimator, and different versions of $(h-h/2)$-based error estimators. In numerical experiments, we use these estimators to steer h-adaptive algorithms, and compare the effectivity of the different approaches.

1. Introduction

The FEM–BEM coupling is often used for interface problems in unbounded domains, where, e.g. non-linearities are present in a bounded domain and the material is isotropic in the exterior, cf. [22,24,36,34]. The symmetric FEM–BEM coupling was proposed and analyzed by Costabel [22] and attracted most attention in the mathematical literature. In engineering, however, more often the coupling procedure proposed by Johnson and Nédélec [37] is used since it only involves two integral operators instead of four. Only very recently, Sayas [46] proved that the Johnson–Nédélec coupling is well-posed even on polygonal domains, whereas numerical evidence of this was already known for many years, cf. e.g. [23].

To the best of our knowledge, the numerical analysis of a posteriori FEM–BEM error estimators has only been derived for the symmetric coupling. Most of the results follow the concept of two-level error estimation introduced in [42], see also the recent work [39] and the references therein. Other approaches include residual-based error estimators which have first been studied in [20], and recently also $(h-h/2)$-based error estimators [5].

In this work, we transfer these three classes of a posteriori error estimators from the symmetric coupling to the Johnson–Nédélec coupling. As model problem serves, for the ease of presentation, the interface problem for the Laplacian in two dimensions with an inhomogeneous volume force in the interior. We then formulate adaptive mesh-refining algorithms for each of these three approaches. In numerical experiments, we finally compare the effectiveness.

The detailed outline of this work reads as follows: In Section 2.1, we state our model problem and fix the notation of the integral operators involved. Section 2.2 introduces the Galerkin discretization and sketches the result of Sayas [46]. For some implementational reasons, we also discretize the given boundary data to which integral operators are applied. This allows to work with discrete integral operators, i.e. matrices, in the implementation and leads to some perturbed Galerkin formulation given in Section 2.3.

Section 3 is the heart of this work and contains the a posteriori error analysis. First, we collect the necessary notation in Sections 3.1 and 3.2. The a posteriori error control of the approximation error for the boundary data is discussed in Section 3.3. In Section 3.4, we study the residual error estimator ϱ_ℓ from [20]. In Section 3.5, we recall the $(h-h/2)$-error estimator μ_ℓ from [5] and discuss the so-called saturation assumption, whereas Section 3.6 is concerned with the two-level error estimator τ_ℓ from [42]. With certain modifications of the analysis from [5,20,42], we transfer these error estimators from the symmetric coupling to the Johnson–Nédélec coupling and can formulate and prove the according results. However, we stress that, first, our version of ϱ_ℓ is improved in the sense that it involves volume oscillations instead of the volume residual terms and, second, we also prove global equivalence $\mu_\ell \simeq \tau_\ell$ of $(h-h/2)$- and two-level error estimator. Finally, a short Section 3.7 provides local relations of τ_ℓ and ϱ_ℓ.

Section 4 considers an experiment from the literature for which uniform and adaptive mesh-refinement are compared with

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doi:10.1016/j.enganabound.2011.07.017

respect to empirical convergence rate and computational time. Finally, we conclude our work in Section 5 with an overview on the analytical and numerical results of this paper. Moreover, we state possible generalizations of our results for 3D problems and pose some questions for further research.

2. Johnson–Nédélec coupling

2.1. Model problem

We consider the linear interface problem

$$\begin{cases} -\Delta u^{\rm int} = f \quad \text{in } \Omega^{\rm int} := \Omega, \\ -\Delta u^{\rm ext} = 0 \quad \text{in } \Omega^{\rm ext} := \mathbb{R}^2 \backslash \overline{\Omega}, \\ u^{\rm int} - u^{\rm ext} = u_0 \quad \text{on } \Gamma, \\ \partial_n u^{\rm int} - \partial_n u^{\rm ext} = \phi_0 \quad \text{on } \Gamma, \\ u^{\rm ext}(x) = \mathcal{O}(|x|^{-1}) \quad \text{as } |x| \to \infty. \end{cases} \tag{1}$$

Here, Ω is a bounded Lipschitz domain in $\mathbb{R}^2$ with boundary $\Gamma := \partial\Omega$ and exterior unit normal vector n. The given data satisfy $f \in L^2(\Omega)$, $u_0 \in H^{1/2}(\Gamma)$, and $\phi_0 \in H^{-1/2}(\Gamma)$. The space $H^{1/2}(\Gamma)$ is precisely the space of all traces of functions from $H^1(\Omega)$, and $H^{-1/2}(\Gamma)$ is the dual of $H^{1/2}(\Gamma)$ with respect to the extended $L^2(\Gamma)$-scalar product. To guarantee the solvability of (1), we need the data to satisfy $\langle \phi_0, 1 \rangle_\Gamma + \langle f, 1 \rangle_\Omega = 0$. As usual, (1) is understood in the weak sense, and the sought solutions satisfy $u^{\rm int} \in H^1(\Omega)$ and $u^{\rm ext} \in H^1_{\ell oc}(\Omega^{\rm ext}) = \{v : \Omega^{\rm ext} \to \mathbb{R} : \forall K \subset \Omega^{\rm ext} \text{ compact } v \in H^1(K)\}$ with $\nabla u^{\rm ext} \in L^2(\Omega^{\rm ext})$.

Problem (1) is equivalently stated via the Johnson–Nédélec FEM–BEM coupling proposed in [37]: Find $\mathbf{u} := (u, \phi) \in \mathcal{H} := H^1(\Omega^{\rm int}) \times H^{-1/2}(\Gamma)$ such that

$$\langle \nabla u, \nabla v \rangle_\Omega - \langle \phi, v \rangle_\Gamma = \langle f, v \rangle_\Omega + \langle \phi_0, v \rangle_\Gamma \quad \text{for all } v \in H^1(\Omega^{\rm int}),$$

$$\langle \psi, (\tfrac{1}{2} - \mathfrak{K}) u + \mathfrak{V}\phi \rangle_\Gamma = \langle \psi, (\tfrac{1}{2} - \mathfrak{K}) u_0 \rangle_\Gamma \quad \text{for all } \psi \in H^{-1/2}(\Gamma). \tag{2}$$

Here, $\mathfrak{V}$ denotes the simple-layer potential and $\mathfrak{K}$ denotes the double-layer potential. With

$$G(z) := -\frac{1}{2\pi} \log|z| \quad \text{for } z \in \mathbb{R}^2 \backslash \{0\} \tag{3}$$

the fundamental solution of the 2D Laplacian, these integral operators formally read for $x \in \Gamma$ as follows:

$$(\mathfrak{V}\psi)(x) = \int_\Gamma G(x-y)\psi(y)\, d\Gamma(y), \tag{4}$$

$$(\mathfrak{K}v)(x) = \int_\Gamma \partial_{n(y)} G(x-y) v(y)\, d\Gamma(y). \tag{5}$$

By continuous extension, these definitions provide linear boundary integral operators $\mathfrak{V} \in L(H^{-1/2}(\Gamma); H^{1/2}(\Gamma))$ and $\mathfrak{K} \in L(H^{1/2}(\Gamma); H^{1/2}(\Gamma))$. By scaling of Ω, we may assume that $\mathrm{diam}(\Omega) < 1$ to ensure the uniform ellipticity of $\mathfrak{V}$, i.e.

$$\|\psi\|^2_{H^{-1/2}(\Gamma)} \lesssim \langle \psi, \mathfrak{V}\psi \rangle_\Gamma \quad \text{for all } \psi \in H^{-1/2}(\Gamma).$$

In particular, $\langle \phi, \psi \rangle_{\mathfrak{V}} := \langle \phi, \mathfrak{V}\psi \rangle_\Gamma$ is a scalar product, and

$$\|\psi\|^2_{\mathfrak{V}} := \langle \psi, \mathfrak{V}\psi \rangle_\Gamma \quad \text{for } \psi \in H^{-1/2}(\Gamma)$$

defines an equivalent norm on $H^{-1/2}(\Gamma)$. The reader is referred to e.g. [41] for proofs and further details on these integral operators. The link between (1) and (2) is provided by $u = u^{\rm int}$ and $\phi = \partial_n u^{\rm ext}$, and $u^{\rm ext}$ is then given by the third Green's formula

$$u^{\rm ext}(x) = \tilde{\mathfrak{K}}(u - u_0)(x) - \tilde{\mathfrak{V}}\phi(x) \quad \text{for } x \in \Omega^{\rm ext}, \tag{6}$$

where the potentials $\tilde{\mathfrak{V}}$ and $\tilde{\mathfrak{K}}$ formally denote the operators $\mathfrak{V}$ and $\mathfrak{K}$, but are now evaluated in $\Omega^{\rm ext}$ instead of Γ. Note carefully that we do not use a notational difference for the function $u \in H^1(\Omega)$ and its trace $u \in H^{1/2}(\Gamma)$, for which we compute the boundary integral $(\frac{1}{2} - \mathfrak{K})u$ in (2).

We stress that the second equation of the Johnson–Nédélec FEM–BEM coupling (2) is the same as for the mathematically well-studied symmetric coupling. It has already been proved in [37] that problem (2) is well-posed on the continuous level, i.e. it admits a unique solution $\mathbf{u} = (u, \phi) \in \mathcal{H}$.

2.2. Galerkin discretization

Let $\mathcal{T}_\ell$ be a regular triangulation of Ω into triangles $T_j \in \mathcal{T}_\ell$ and $\mathcal{E}^\Gamma_\ell$ a partition of the coupling boundary Γ into piecewise affine line segments $E_j \in \mathcal{E}^\Gamma_\ell$. Throughout, the index $\ell \in \mathbb{N}_0$ indicates the current step of the adaptive loop considered below. We use a conforming discretization with continuous and $\mathcal{T}_\ell$-piecewise affine finite elements in Ω and $\mathcal{E}^\Gamma_\ell$-piecewise constants on Γ, i.e. the discrete spaces read

$$\mathcal{X}_\ell := \mathcal{S}^1(\mathcal{T}_\ell) \times \mathcal{P}^0(\mathcal{E}^\Gamma_\ell) \subset H^1(\Omega) \times H^{-1/2}(\Gamma) = \mathcal{H}. \tag{7}$$

We stress that our analysis does not enforce any coupling of $\mathcal{E}^\Gamma_\ell$ and $\mathcal{T}_\ell$. However, for the ease of presentation and implementation, we will assume throughout that the boundary mesh $\mathcal{E}^\Gamma_\ell = \mathcal{T}_\ell|_\Gamma$ is obtained by restriction of the triangulation $\mathcal{T}_\ell$ to the boundary Γ.

The Galerkin formulation of (2) then reads as follows: Find $\mathbf{U}^\star_\ell := (U^\star_\ell, \Phi^\star_\ell) \in \mathcal{X}_\ell$ such that

$$\langle \nabla U^\star_\ell, \nabla V_\ell \rangle_\Omega - \langle \Phi^\star_\ell, V_\ell \rangle_\Gamma = \langle f, V_\ell \rangle_\Omega + \langle \phi_0, V_\ell \rangle_\Gamma,$$

$$\langle \Psi_\ell, (\tfrac{1}{2} - \mathfrak{K}) U^\star_\ell + \mathfrak{V}\Phi^\star_\ell \rangle_\Gamma = \langle \Psi_\ell, (\tfrac{1}{2} - \mathfrak{K}) u_0 \rangle_\Gamma \tag{8}$$

for all $\mathbf{V}_\ell := (V_\ell, \Psi_\ell) \in \mathcal{X}_\ell$. Only very recently [46, Theorem 2], it has been proven that the discrete formulation (8) is well-posed and admits a unique Galerkin solution $\mathbf{U}^\star_\ell \in \mathcal{X}_\ell$. We stress that the following result applies, in particular, also to the continuous formulation (2) and provides an alternate proof for the existence and uniqueness of a solution of the Johnson–Nédélec FEM–BEM coupling.

Proposition 1 (*Sayas [46]*). *Suppose that X_ℓ is a closed subspace of $H^1(\Omega)$ and Y_ℓ is a closed subspace of $H^{-1/2}(\Gamma)$ which satisfy*

$$1 \in X_\ell \quad \text{as well as} \quad 1 \in Y_\ell, \tag{9}$$

i.e. the discrete spaces contain the constant functions. With $\mathcal{X}_\ell := X_\ell \times Y_\ell$, the linear operator $\mathbb{H} : \mathcal{X}_\ell \to \mathcal{X}^*_\ell$

$$(\mathbb{H}\mathbf{U}_\ell)(\mathbf{V}_\ell) := \langle \nabla U_\ell, \nabla V_\ell \rangle_\Omega - \langle \Phi_\ell, V_\ell \rangle_\Gamma + \langle \Psi_\ell, (\tfrac{1}{2} - \mathfrak{K}) U_\ell + \mathfrak{V}\Phi_\ell \rangle_\Gamma \tag{10}$$

for $\mathbf{U}_\ell = (U_\ell, \Phi_\ell)$, $\mathbf{V}_\ell = (V_\ell, \Psi_\ell) \in \mathcal{X}_\ell$ defines an isomorphism, where the bounds of the operator norms $\|\mathbb{H}\|$ and $\|\mathbb{H}^{-1}\|$ depend only on Ω, but not on the chosen spaces X_ℓ and Y_ℓ. In particular, the variational form (8) admits a unique solution $\mathbf{U}^\star_\ell \in \mathcal{X}_\ell$. Moreover, there holds the Céa-type quasi-optimality

$$|||\mathbf{u} - \mathbf{U}^\star_\ell||| \le C_{\rm opt} \min_{\mathbf{V}_\ell \in \mathcal{X}_\ell} |||\mathbf{u} - \mathbf{V}_\ell||| \tag{11}$$

with $|||\mathbf{v}|||^2 := \|v\|^2_{H^1(\Omega)} + \|\psi\|^2_{\mathfrak{V}}$ for $\mathbf{v} = (v, \psi) \in \mathcal{H}$, and the constant $C_{\rm opt} > 0$ depends only on Ω, but not on $\mathcal{X}_\ell$ or the given data f, ϕ_0, and u_0.

2.3. Perturbed Galerkin discretization

The right-hand side of (8) involves the evaluation of $\mathfrak{K}u_0$, which can be computed by methods proposed in [19,44,45]. In this work, we will follow another approach. We propose to approximate at least the given trace data $u_0 \in H^{1/2}(\Gamma)$ by appropriate discrete functions. One reason for this is that so-called *fast methods for boundary integral operators* usually deal with discrete