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Generalized boundary element method for galerkin boundary integrals

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Abstract

A meshless approach to the Boundary Element Method in which only a scattered set of points is used to approximate the solution is presented. Moving Least Square approximations are used to build a Partition of Unity on the boundary and then used to construct, at low cost, trial and test functions for Galerkin approximations. A particular case in which the Partition of Unity is described by linear boundary element meshes, as in the Generalized Finite Element Method, is then presented. This approximation technique is then applied to Galerkin boundary element formulations. Finally, some numerical accuracy and convergence solutions for potential problems are presented for the singular, hypersingular and symmetric approaches.

Keywords: Galerkin method; BEM p-adaptivity; Symmetrical BEM; Hp-Clouds; Partition of unity

1. Introduction

In the last decade a number of meshless procedures have been proposed in the FEM community. These include: The Smoothed Particle Hydrodynamics Method, The Diffuse Element Method [1], Wavelet Galerkin Method [2], The Element Free Galerkin Method, (EFGM), [3], Reproducing Kernel Particle Method (RKPM) [4], The Meshless Local Petrov–Galerkin Method [5], the Natural Element Method [6], Partition of Unity Method [7], and the hp-Cloud Methods e.g. [8,9]. The latter has the further appeal of naturally introducing a procedure for performing hp-adaptivity, in a very flexible way, avoiding the construction of functions by sophisticated hierarchical techniques. The advantages of these procedures are, however, balanced by increased computational cost since a mesh is still needed for integration purposes and, at each integration point, the Partition of Unity must be computed since the covering of each point is arbitrary. The cost can be reduced by using a linear Lagrangian Partition of Unity as in the Finite Element Method as proposed by Oden, Duarte and Zienkiewicz [10] and later denoted by the Generalized Finite Element Method [11], (GFEM), which can be understood as a Generalization of the Partition of Unity Method [7]. More recently, Sukumar and his co-workers [12], proposed the Extended Finite Element Method, (XFEM), which presents similar characteristics as the GFEM.

The meshless procedures have also attracted the attention of an increasing number of researchers within the Boundary Element community. Among many contributions, we may cite the Boundary Node Method [13–15], Local Boundary Integral Equation [16,17], Boundary Particle Method [18], Radial Point Interpolation Meshless Method (Radial PIM) [19–22], and Boundary Cloud Method (BCM) [23]. Most of the meshless methods use approximation functions along the lines of the Moving Least Squares Method [24] and of the EFGM.

The present work is an extension of the hp-Cloud Method in order to apply it to the Boundary Element Method, following the path presented in [25].

Hp-Cloud approximations have been proved to be more efficient than those of the EFGM, [9], [26], and for this reason they were used in [25]. Later, Oden, Duarte and

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Zienkiewicz, [10], proposed that, instead of using circles or rectangles for defining the Clouds around each node, it would be more convenient to use linear finite element meshes. Here the Clouds associated to node 'i' would be built by the union of the 'elements' connected to this node. This concept greatly reduces the number of floating point operations, since the Partition of Unity is known beforehand and allows standard integration routines for integrating the nodal matrices. This new scheme led to the Generalized Finite Element Method, GFEM.

In this paper, some choices of Partition of Unity are discussed and one of them is selected to be applied to the Galerkin Boundary Element Method. This Partition of Unity is then enriched by a set of functions like polynomials of equal or unequal degrees in different directions, particular solutions, or other reasonable functions to span the approximation space. A choice of error indicators in order to adaptively enrich the Partition of Unity is here described. This new technique is hereafter called the Generalized Galerkin Boundary Element Method (GGBEM). The L-shaped domain and the Motz potential problems are solved by the Classic (singular), Hyper and Symmetric methods and their results for both uniform and adaptive enrichment are compared and discussed.

The remainder of this paper is outlined as follows: Section 2 summarizes the Galerkin boundary integral equations for potential 2D problems; Section 3 describes the main topics of the Moving Least Squares Method, MLSM; Section 4 presents the hp-Cloud Partition of Unity functions and their enrichment is described in Sections 5 and 6 discusses some of the possible MLSM weighting functions and one in particular which leads to the generalized formulations; Section 7 presents an error indicator for the Galerkin boundary integral equations; Section 8 summarizes the selected integration and regularization procedures; Section 9 presents results of the proposed formulation for the L-shaped domain and the Motz potential problems; and the conclusions are given in Section 10.

2. Galerkin boundary elements

Since this work is mainly focused on the numerical characteristics of the approximation method, a simple differential equation in two dimensions is dealt with here.

Let us define a domain $\Omega \subset \Re^2$ by a Lipschitz boundary $\Gamma = \Gamma_{\mathcal{D}} \cap \Gamma_{\mathcal{N}}$, where the Dirichlet, $\Gamma_{\mathcal{D}}$, and Neumann, $\Gamma_{\mathcal{N}}$, parts of the boundary have null intersection, $\Gamma_{\mathcal{D}} \cap \Gamma_{\mathcal{N}}=$. Equilibrium is stated by the Laplace equation with Dirichlet and Neumann boundary conditions

$$-\Delta T(\mathbf{x}) = 0 \text{ on } \Omega, \quad T(\mathbf{x}) = f \text{ on } \Gamma_{\mathcal{D}},$$
$$\frac{\partial T}{\partial n} = g \text{ on } \Gamma_{\mathcal{N}}, \tag{1}$$

where by T we denote the unknown potential field and by $\partial T/\partial n$ its normal derivative.

The Galerkin or Variational approach in boundary integral equations [27] is given by

$$c_1 \int_\Gamma \varphi_j(d)\varphi_k(d)\mathrm{d}\Gamma(d)T_j = \int_\Gamma\int_\Gamma G(\mathbf{x},\mathbf{d})\varphi_i(\mathbf{x})\varphi_k(\mathbf{d})\mathrm{d}\Gamma(\mathbf{x})\mathrm{d}\Gamma(\mathbf{d})\frac{\partial T_i}{\partial n} - \int_\Gamma\int_\Gamma \frac{\partial G(\mathbf{x},\mathbf{d})}{\partial n(\mathbf{x})}\varphi_i(\mathbf{x})\varphi_k(\mathbf{d})\mathrm{d}\Gamma(\mathbf{x})\mathrm{d}\Gamma(\mathbf{d})T_i \tag{2}$$

and also by an analogous expression for the normal derivative,

$$c_2 \int_\Gamma \varphi_j(\mathbf{d})\varphi_k(\mathbf{d})\mathrm{d}\Gamma(\mathbf{d})\frac{\partial T_j}{\partial n} = \int_\Gamma\int_\Gamma \frac{\partial G(\mathbf{d},\mathbf{x})}{\partial n(\mathbf{d})}\varphi_i(\mathbf{x})\varphi_k(\mathbf{d})\mathrm{d}\Gamma(x)\mathrm{d}\Gamma(\mathbf{d})\frac{\partial T_i}{\partial n} - \int_\Gamma\int_\Gamma \frac{\partial^2 G(\mathbf{d},\mathbf{x})}{\partial n(\mathbf{d})\partial n(\mathbf{x})}\varphi_i(\mathbf{x})\varphi_k(\mathbf{d})\mathrm{d}\Gamma(\mathbf{x})\mathrm{d}\Gamma(\mathbf{d})T_i \tag{3}$$

In these expressions, φ_s are the test and trial functions, c_1 and c_2 are constants determined from the *Jump Term,* **d** and **x** are the source and field point locations and G is a fundamental solution. The set of algebraic equations obtained from expression (2) is the starting point of the classical Galerkin approach. When Eq. (3) is used, an alternative set of equations is obtained, usually called the Hypersingular Galerkin approach. The Symmetric Galerkin approximation results from a choice of equations from both previous sets. In this work these approaches are, respectively, denoted by *Classic*, *Hyper* and *Symmetric*. The characteristics of the approximation space as well as the methodology of construction of the approximation functions is the focus of the next section.

3. The moving least squares method applied to the cloud method

The Moving Least Square Method (MLS) [24], is a generalization of the conventional Least Squares Method and has the important property of allowing us to weight, in different forms, the information at arbitrarily placed points in the domain. The next paragraphs present a brief description of the method.

Let a body occupying a domain $\Omega \in \Re^n$, $n=1$, 2, or 3,with contour Γ, and let f_α, $\alpha=0,1,2,3,\ldots N$, be