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Spectral methods based on new formulations for coupled Stokes and Darcy equations

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ABSTRACT

In this paper we consider the numerical solution of the Stokes and Darcy coupled equations, which frequently appears in porous media modeling. The main contribution of this work is as follows: First, we introduce a new formulation for the Stokes/Darcy coupled equations, subject respectively to the Beavers–Joseph–Saffman interface condition and an alternative matching interface condition. Secondly, we prove the well-posedness of these weak problems by using the classical saddle point theory. Thirdly, some spectral approximations to the weak problems are proposed and analyzed, and some error estimates are provided. It is found that the new formulations significantly simplify the error analysis and numerical implementation. Finally, some two-dimensional spectral and spectral element numerical examples are provided to demonstrate the efficiency of our methods.

1. Introduction

The model of Stokes equations coupled with Darcy equations has become a very active research area during the last decade. The extensive investigation of this model has been motivated by a variety of applications, including bio-engineering [20,23], environment research [8], industry [19,21,25], and also by the increasing need of simpler, more accurate, and more efficient numerical method to solve it.

First, one of the most important things dealing with the Stokes and Darcy coupled equations is to find the correct condition on the interface, which separates the fluid flow and the porous media subdomains. Based on the original experimental work of Beavers and Joseph on the coupling conditions between free and porous flow, a so-called Beavers–Joseph–Saffman interface condition and several formulations associated to this condition have been investigated in a number of papers, see for example [22,27]. These conditions are derived according to mass conservation, balance of normal forces, and the Beavers–Joseph–Saffman law. On the other side, there also exist alternative interfacial conditions for different coupled equations, such as the Navier–Stokes/Euler coupling [36]. One of the main work of this paper is to establish suitable variational forms for the Stokes/Darcy coupled equations, subject to different interface conditions. Thanks to the good properties of these weak forms, which are some kind of saddle point problems, the existence and uniqueness of the solution for the Stokes/Darcy coupled problems can be proved. Furthermore, as we are going to see, these variational forms significantly simplify the design and analysis of the numerical method.

Secondly, concerning the existing numerical methods in the literature for the Stokes/Darcy coupled equations, we mention, among others, the finite element methods [1,2,6,9,12,13,18,28,29,33,35] based on appropriate combinations of stable

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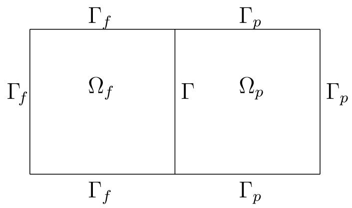

Fig. 1. The model computational domain of the coupled problem.

elements for free flow and porous flow, mortar finite techniques [14,15], finite volume method [7], and discontinuous Galerkin method [16]. In this work, we consider a spectral element approximation to the Stokes/Darcy coupled equations. Although other kinds of numerical methods can also be considered, the spectral method would be found useful in some practical applications, especially when the problem under consideration admits smooth solutions. As we know, the spectral element method has the fascinating merit of the high accuracy and great flexibility in dealing with geometrically complex domains. For this reason we will focus on the spectral element method for the numerical approximation of the Stokes/Darcy coupled equations. An error analysis of this method is carried out, and a quasi-optimal error estimate is obtained.

Thirdly, our spectral element method for the Stokes/Darcy coupled equations is based on the weak formulations we mentioned above, and leads to discrete saddle point algebraic systems with some particular properties. Several algorithms are possible to solve the discrete saddle point system generated by the spectral element discretization. One possible choice is to use a direct method, which consists in solving the nodal pressure and velocity unknowns in a coupled form [34,38]. Another possibility is to replace the continuity equations with a Poisson-like equation for the pressure [17,24]. This approach requires a discretization of the continuous pressure problem, and pressure boundary conditions must be supplied. In this paper, the so-called Uzawa algorithm is employed to solve the final discrete saddle point problem. The algorithm consists in applying block Gaussian elimination and back-substitution for pressure and velocity yielding two positive definite symmetric systems. It has the following advantages: (i) among numerous algorithms, the Uzawa decoupling procedure is more attractive in terms of computational complexity and memory requirement than a direct algorithm since the solution process is completely decoupled for the pressure and velocity; (ii) the block diagonal structure in the velocity system allows us to inverse the velocity matrix easily, and therefore reduce the cost in each iteration for the pressure calculation.

The rest of this paper is organized as follows: In Section 2, we first present an overview of the existing models about the Stokes/Darcy coupled equations, then propose our new formulation under Beavers–Joseph–Saffman interface condition. We discuss in Section 3 an alternative interface condition, stemming from the classical Navier–Stokes/Euler coupling, and show that suitable weak formulation exists for this new condition. In Section 4, we first establish the well-posedness of these weak problems by using the standard saddle point theory. Then basing on this result we propose the spectral approximation to the coupled problems. The well-posedness of the discrete problems and error estimates for the numerical solutions are derived. Section 5 gives some implementation details of the Uzawa algorithm, together with a series of numerical examples to support our theoretical results. Finally, we summarize and prospect our research in the last section.

Throughout the rest of the paper we utilize the standard terminology for Sobolev spaces. In particular, if D is a bounded connected domain and $r \in R$, then $|\cdot|_{r,D}$ and $\|\cdot\|_{r,D}$ stand for the seminorm and norm in the Sobolev spaces $H^r(D)$, $[H^r(D)]^2$, and $[H^r(D)]^{2\times 2}$. Also, we use C and c, with or without subscripts, bars, tildes or hats, to mean generic positive constants independent of the discretization parameters, which may not be the same at different occurrences.

2. Formulations of the Stokes/Darcy coupled equations

2.1. Modeling equations

We are interested in the Stokes/Darcy coupled equations in two dimensions. We assume that Ω is an open bounded subset of R^2, with Lipschitz boundary $\partial\Omega$. Ω_f and Ω_p are respectively the fluid and porous media subdomains of Ω, such that $\Omega_f \cap \Omega_p = \emptyset$, $\bar{\Omega}_f \cup \bar{\Omega}_p = \bar{\Omega}$. We denote $\Gamma = \partial\Omega_f \cap \partial\Omega_p$, $\Gamma_f = \partial\Omega_f \cap \partial\Omega$, and $\Gamma_p = \partial\Omega_p \cap \partial\Omega$. The unit normal vector $\mathbf{n}_f$ on the boundary Γ_f is chosen pointing outwards from Ω_f (similarly for the notation $\mathbf{n}_p$). We use the notation $\boldsymbol{\tau}$ to mean a unit tangent vector on the boundary Γ. A model computation domain is shown in Fig. 1. For the sake of simplification, the description and analysis of our method will be based on this model domain, however it is worthwhile to emphasize that the results obtained hereafter for the model domain can be naturally generalized to more complex configuration without essential difficulty.

We denote by $\mathbf{u} = (\mathbf{u}_f, \mathbf{u}_p)$ the fluid velocity and by $p = (p_f, p_p)$ the fluid pressure, where $\mathbf{u}_i = \mathbf{u}|_{\Omega_i}$ and $p_i = p|_{\Omega_i}$, $i = f, p$. The flow in the domain Ω_f is assumed to be of Stokes type, and therefore its governing equations read:

$$\begin{cases} -\nabla\cdot\big(-p_f I + 2\nu D(\mathbf{u}_f)\big) = \mathbf{f} & \text{in } \Omega_f, \\ \nabla\cdot\mathbf{u}_f = 0 & \text{in } \Omega_f, \\ \mathbf{u}_f = 0 & \text{on } \Gamma_f, \end{cases} \tag{2.1}$$