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Stable multi-domain spectral penalty methods for fractional partial differential equations

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ABSTRACT

We propose stable multi-domain spectral penalty methods suitable for solving fractional partial differential equations with fractional derivatives of any order. First, a high order discretization is proposed to approximate fractional derivatives of any order on any given grids based on orthogonal polynomials. The approximation order is analyzed and verified through numerical examples. Based on the discrete fractional derivative, we introduce stable multi-domain spectral penalty methods for solving fractional advection and diffusion equations. The equations are discretized in each sub-domain separately and the global schemes are obtained by weakly imposed boundary and interface conditions through a penalty term. Stability of the schemes are analyzed and numerical examples based on both uniform and nonuniform grids are considered to highlight the flexibility and high accuracy of the proposed schemes.

1. Introduction

The basic idea behind fractional calculus has a history similar to and aligned with that of more classical calculus and the topic has attracted the interests of mathematicians who contributed fundamentally to the development of classical calculus, including L' Hospital, Leibniz, Liouville, Riemann, Grünward, and Letnikov [6]. In spite of this, the development and analysis of fractional calculus and fractional differential equations is less mature than that associated with classical calculus. However, during the last decade it has become increasingly clear that fractional calculus emerges naturally as a tool for the description of a broad range of non-classical phenomena in the applied sciences and engineering [10,17,29]. A striking example of this is a model for anomalous transport processes and diffusion, leading to fractional partial differential equations [5,23,24], but other examples are readily available for the modeling of frequency dependent damping behavior of many viscoelastic materials [12,2,3], continuum and statistical mechanics [22], solid mechanics [28], and economics [4].

Models involving fractional derivatives can be divided into two major types: time fractional differential equations, typically associated with phenomena with memory or non-Markovian processes and spatial fractional partial differential equations (FPDE), often used to model anomalous diffusion or dispersion with enhanced diffusion speed (also called super-diffusion) [26]. With an expanding range of applications and models based on fractional calculus comes a need for the development of robust and accurate computational methods for solving these equations. For the time fractional problems, there is a substantial number of publications on a variety of numerical schemes [18,19,9,36–38]. For the spatial FPDEs, publications on the numerical schemes are relatively sparse, and the majority of the publications are based on finite difference methods of order one or at most two [8,21,25,26,30,31,33–35]. Zhou [39] recently proposed a compact finite difference

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scheme of third order accuracy, albeit with strict limitation on the regularity of the boundary. In addition, finite element methods of order $\mu (1 < \mu < 2)$ has also been used for space and time fractional Fokker–Planck equation in [11].

Solutions of fractional diffusion problem are generally endowed with substantial smoothness, suggesting that higher order accurate global methods may be attractive alternatives to more traditional techniques yet there appears to be very limited work in this direction. In this work we propose a flexible high order multi-domain spectral penalty method for the space FPDE, working both one uniform and nonuniform grids. The method takes it motivation from classic spectral methods as discussed in [15]. However, in contrast to the classic work, we need to generalize the notation of a fractional differential matrix. With the spectral method developed for a single domain, we extend it to the multi-domain case to allow flexibility and non-uniform resolution. This requires a construction of a differential matrix on multi-domains to reflect the global operator. In this paper, we construct a *hp* approximation to the fractional derivative on multi-domains based on Jacobian polynomial and analyze the approximation accuracy of the method. Based on the fractional differential multi-domain matrix, we propose a stable spectral multi-domain approach based on weakly imposed boundary conditions through a penalty term [13,14]. The outcome is a flexible and accurate family of schemes that can be applied to the modeling of fractional partial differential equations of arbitrary order and we shall demonstrate its application to fractional advection and diffusion problems.

What remains is organized as follows. In Section 2 we offer some background material on fractional derivatives with a particular focus on the Caputo definition. In Section 3 we introduce Jacobi polynomials and use these to express the discrete approximation of the fractional derivative through differentiation matrices and discuss the approximation properties of this formulation. This sets the stage for Section 4 where we propose multi-domain methods for solving fractional advection and diffusion problems and establish the stability of these penalty schemes. We also illustrate the performance of the schemes before concluding in Section 5 with a brief overview and outlook.

2. Fractional calculus, definitions, and basic properties

A complication associated with fractional derivatives is that there are several definitions of exactly what a fractional derivative means [27]. If we consider the function $f(x) \in C^n[0,b]$, then the Riemann–Liouville fractional derivative of order α $(n-1 < \alpha \leqslant n, n \in \mathbb{N})$ is of the form,

$$ {}^{RL}_{0}D^{\alpha}_{x} f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x \frac{f(\tau)}{(x-\tau)^{\alpha+1-n}}\, d\tau, & n-1 < \alpha < n, \\ f^{(n)}(x), & \alpha = n. \end{cases} \tag{1} $$

An alternative definition, known as the Caputo fractional derivative, is defined as

$$ {}_{0}D^{\alpha}_{x} f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha+1-n}}\, d\tau, & n-1 < \alpha < n, \\ f^{(n)}(x), & \alpha = n. \end{cases} \tag{2} $$

These two definitions are not generally equivalent but they are related as

$$ {}_{0}D^{\alpha}_{x} f(x) = {}^{RL}_{0}D^{\alpha}_{x} f(x) - \sum_{\mu=0}^{n-1} \frac{x^{\mu-\alpha} f^{(\mu)}(0)}{\Gamma(\mu+1-\alpha)}. \tag{3} $$

One easily realizes the equivalence between the two for $f^{(\mu)}(0) = 0$, $\mu = 0, \dots, n-1$. The Caputo definition is often preferred when considered in the context of differential equations since it allows for imposing initial and boundary conditions on classic derivatives. For the Riemann–Liouville definition, such conditions must be imposed on fractional derivatives which is often not available. For this reason we shall focus on the Caputo definition in this work.

For the Caputo derivative, we have the following important properties [27],

Lemma 2.1.

$$ {}_{0}D^{\alpha}_{x} C = 0, \quad \alpha > 0,\ C \text{ is a constant.} \tag{4} $$

$$ {}_{0}D^{\alpha}_{x}\big(f(x) + g(x)\big) = {}_{0}D^{\alpha}_{x} f(x) + {}_{0}D^{\alpha}_{x} g(x). \tag{5} $$

$$ {}_{0}D^{\alpha}_{x} x^{\gamma} = \begin{cases} 0, & \alpha > \gamma, \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}, & 0 < \alpha \leqslant \gamma. \end{cases} \tag{6} $$

At times, (1) and (2) are referred to as the left Riemann–Liouville derivative and left Caputo derivative, respectively. In a similar fashion, one can likewise define the right Riemann–Liouville and Caputo derivative as