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A domain decomposition method of stochastic PDEs: An iterative solution techniques using a two-level scalable preconditioner ☆

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ABSTRACT

Recent advances in high performance computing systems and sensing technologies motivate computational simulations with extremely high resolution models with capabilities to quantify uncertainties for credible numerical predictions. A two-level domain decomposition method is reported in this investigation to devise a linear solver for the large-scale system in the Galerkin spectral stochastic finite element method (SSFEM). In particular, a two-level scalable preconditioner is introduced in order to iteratively solve the large-scale linear system in the intrusive SSFEM using an iterative substructuring based domain decomposition solver. The implementation of the algorithm involves solving a local problem on each subdomain that constructs the local part of the preconditioner and a coarse problem that propagates information globally among the subdomains. The numerical and parallel scalabilities of the two-level preconditioner are contrasted with the previously developed one-level preconditioner for two-dimensional flow through porous media and elasticity problems with spatially varying non-Gaussian material properties. A distributed implementation of the parallel algorithm is carried out using MPI and PETSc parallel libraries. The scalabilities of the algorithm are investigated in a Linux cluster.

1. Introduction

In the numerical modeling of many practical engineering and natural systems, the random heterogeneities and imperfections of the propagating media may significantly influence the predictive capabilities of computer simulations. Recent advances of high performance computing and sensing technologies motivate computational simulations with extremely high resolution which should integrate efficient uncertainty quantification methods for realistic and credible numerical predictions. For such extreme scale simulations, uncertainty quantification using the standard Monte Carlo simulation may be time-consuming or impractical. To reduce the computational cost of the traditional Monte Carlo simulation, a multi-level Monte Carlo method has recently been proposed [2,3]. Alternatives to Monte Carlo sampling techniques are the spectral stochastic finite element method (SSFEM) whose implementations can exploit non-intrusive collocation method (e.g. [4–11]) or intrusive Galerkin approach (e.g. [12–21]). This investigation focuses on an efficient distributed implementation of Galerkin scheme in SSFEM for large-scale systems, circumventing the need of any random or deterministic sampling. For a comprehensive review of SSFEM, we refer to e.g. [12,14,22–26].

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In the so-called intrusive SSFEM (e.g. [12]), a Galerkin projection scheme transforms the original stochastic linear system into a coupled set of block linear systems. Depending on the scales of random fluctuations present on the system parameters and degree of nonlinearities between the input and output processes, the order of the resulting deterministic linear system can increase significantly in the SSFEM. The computational efficiency of the SSFEM solver is primarily dictated by the solution strategies adopted to tackle the coupled deterministic linear systems. This fact inspires the development of a scalable solver for the SSFEM which can efficiently exploit the modern high performance computing systems.

Preconditioned iterative solvers [27–32] and multigrid methods [33–35,26] have been exploited to develop robust and reliable iterative solvers for the Galerkin projection-based intrusive SSFEM. The block-Jacobi preconditioner based on the mean stiffness matrix is generally employed to accelerate the convergence rate of the iterative methods. However, such iterative solvers equipped with the block-Jacobi preconditioner show performance degradation as the level of uncertainty and order of the deterministic linear system grow in the SSFEM [27,28,30,32,36]. This fact prompts further research to enhance the robustness and performance of iterative techniques in order to solve the large-scale system in the intrusive SSFEM.

To this end, a non-overlapping domain decomposition algorithm is reported in [37] using the intrusive SSFEM. The algorithm relies on a Schur complement-based domain decomposition in the physical space and a polynomial expansion of stochastic processes representing the system parameters and solution field. Extending the basic mathematical formulation in [37], a one-level iterative substructuring technique is presented in [38,39] for efficient solution of the large-scale linear system in the intrusive SSFEM. Numerical experiments demonstrate that the convergence rate of the iterative algorithm is mainly insensitive to the magnitude of the coefficient of variation (i.e. strength of randomness) of the system parameters and order of stochastic dimension. However, the one-level iterative algorithm in [38,39] shows that the iteration count grows linearly as we increase the number of subdomain partitions.

A two-level domain decomposition preconditioner is described in this paper to improve the efficiency of the iterative substructuring technique in the intrusive SSFEM [38,39]. A dual–primal domain decomposition method for SSFEM is reported elsewhere by the authors [40–43]. In particular, the one-level Neumann–Neumann (NN) preconditioner in [38,39] is complemented by a coarse grid in this paper. A collection of corner nodes at the subdomain interfaces constitute the coarse grid. Consequently, a coarse problem is solved at each iteration to spread information across all the subdomains. This information exchange achieved by the coarse grid makes the preconditioner scalable. The two-level preconditioner may be construed to be a probabilistic extension of the Balancing Domain Decomposition by Constraints (BDDC) [44,45] devised for deterministic PDEs. The parallel performances of the previously developed one-level preconditioner [38,39] and the new two-level preconditioner are contrasted using illustrations from linear elasticity and flow through porous media with spatially varying non-Gaussian material properties. We use PETSc [46] and MPI [47] parallel libraries for the distributed implementation of the algorithm.

2. Schur complement system of the stochastic PDEs

We briefly describe Schur complement-based domain decomposition solver for stochastic PDEs [1,37–39,48,40,41]. For an elementary exposition of the methodology, we consider an elliptic stochastic PDE on a domain Ω with a prescribed homogeneous Dirichlet boundary condition on $\partial\Omega$ as

$$\nabla\cdot\big(\kappa(\mathbf{x},\theta)\nabla u(\mathbf{x},\theta)\big)=f(\mathbf{x}),\quad \mathbf{x}\in\Omega, \tag{1}$$

$$u(\mathbf{x},\theta)=0,\quad \mathbf{x}\in\partial\Omega, \tag{2}$$

where θ denotes an element in the sample space, $u(\mathbf{x},\theta)$ is the solution process, $\kappa(\mathbf{x},\theta)$ represents the diffusion coefficient modeled as a strictly positive random field and $f(\mathbf{x})$ is the deterministic forcing term. The finite element discretization of the above elliptic stochastic PDE yields the following stochastic linear system

$$\mathbf{A}(\theta)\mathbf{u}(\theta)=\mathbf{f}, \tag{3}$$

where $\mathbf{A}(\theta)$ is the stochastic stiffness matrix, $\mathbf{u}(\theta)$ is the random response vector and $\mathbf{f}$ is the external forcing vector. For high resolution models, domain decomposition techniques can be exploited to tackle Eq. (3) on parallel computers [1,37–39,48,40,41].

The computational domain Ω is partitioned into n_s non-overlapping subdomains $\Omega=\bigcup_{s=1}^{n_s}\Omega_s$ with interfaces defined as $\Gamma=\bigcup_{s=1}^{n_s}\Gamma_s$ where $\Gamma_s=\partial\Omega_s\backslash\partial\Omega$ [49–52]. The nodes of each subdomain (Ω_s, $s=1,\ldots,n_s$) are partitioned into two subsets: interior nodes that belong to the given subdomain and interface nodes shared by two or more adjacent subdomains as shown in Fig. 1.

According to this partition, the equilibrium system of a typical subdomain Ω_s is expressed as

$$\begin{bmatrix}\mathbf{A}_{II}^s(\theta) & \mathbf{A}_{I\Gamma}^s(\theta)\\ \mathbf{A}_{\Gamma I}^s(\theta) & \mathbf{A}_{\Gamma\Gamma}^s(\theta)\end{bmatrix}\begin{Bmatrix}\mathbf{u}_I^s(\theta)\\ \mathbf{u}_\Gamma^s(\theta)\end{Bmatrix}=\begin{Bmatrix}\mathbf{f}_I^s\\ \mathbf{f}_\Gamma^s\end{Bmatrix} \tag{4}$$

where $\mathbf{u}_I^s(\theta)$ denotes the unknowns associated with the interior nodes of the subdomain Ω_s and $\mathbf{u}_\Gamma^s(\theta)$ corresponds to the interface unknowns associated with the interface boundary Γ_s.