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Exactly well-balanced discontinuous Galerkin methods for the shallow water equations with moving water equilibrium ☆

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ABSTRACT

Hyperbolic conservation laws with source terms often admit steady state solutions where the fluxes and source terms balance each other. To capture this balance and near-equilibrium solutions, well-balanced methods have been introduced and performed well in many numerical tests. Shallow water equations have been extensively investigated as a prototype example. In this paper, we develop well-balanced discontinuous Galerkin methods for the shallow water system, which preserve not only the still water at rest steady state, but also the more general moving water equilibrium. The key idea is the recovery of well-balanced states, a special source term approximation, and the approximation of the numerical fluxes based on a generalized hydrostatic reconstruction. We also study the extension of the positivity-preserving limiter presented in [40] in this framework. Numerical examples are provided at the end to verify the well-balanced property and good resolution for smooth and discontinuous solutions.

1. Introduction

Hyperbolic systems often involve source terms arising from geometrical, reactive, biological or other considerations. They are often referred as balance laws and have wide applications in different fields including chemistry, biology, fluid dynamics, astrophysics, and meteorology. In one dimension, they usually take the form

$$U_t + f(U,x)_x = s(U,x), \tag{1.1}$$

where U is the solution vector, $f(U,x)$ is the flux and $s(U,x)$ is the source term. Such balance laws admit steady state solutions which are usually non-trivial and often carry important physical meaning. A straightforward numerical scheme may fail to preserve exactly these steady states. The well-balanced schemes are introduced to preserve exactly, at a discrete level, some of these equilibrium solutions. One major advantage of the well-balanced schemes is that they can accurately resolve small perturbations to such steady state solutions with relatively coarse meshes. Indeed, if a scheme cannot balance the effects of fluxes and source terms, it may introduce spurious oscillations near steady state. In order to reduce these, the grid must be refined more than necessary. On the other hand, well-balanced schemes promise to be efficient near steady state.

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A prototype example investigated extensively in the literature is shallow water equations with a non-flat bottom topography, which are used to model flows in rivers and coastal areas, and have wide applications in ocean, hydraulic engineering, and atmospheric modeling. This system describes the flow as a conservation law with an additional source term due to the bottom topography. In one space dimension, it takes the form

$$\begin{cases} h_t + (hu)_x = 0, \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = -ghb_x, \end{cases} \tag{1.2}$$

where h denotes the water height, u is the velocity of the fluid, b represents the bottom topography and g is the gravitational constant. Only the source term due to the bottom topography is taken into account in this system, but other terms could also be added in order to include effects such as friction on the bottom as well as variations of the channel width.

Research on well-balanced numerical methods for the shallow water system has attracted many attentions in the past two decades. Many researchers have developed a large number of well-balanced methods to exactly preserve the still water at rest steady state

$$u = 0 \quad \text{and} \quad h + b = \text{const}, \tag{1.3}$$

see, e.g. [1,2,6,5,14–16,19–21,25,26,34,35], and the recent review paper [23] for more details. Shallow water equations (1.2) also admit the general moving water equilibrium, given by

$$m := hu = \text{const} \quad \text{and} \quad E := \frac{1}{2}u^2 + g(h + b) = \text{const}, \tag{1.4}$$

where m, E are the moving water equilibrium variables. Still water steady state is simply a special case of this, when the velocity reduces to zero. Most well-balanced methods for the still water steady state cannot preserve the moving water equilibrium, and it is significantly more difficult to obtain well-balanced schemes for such equilibrium.

In a recent paper [39], we have shown several numerical examples to demonstrate the advantage of moving-water well-balanced schemes over still-water well-balanced schemes for the shallow water equations. Those numerical examples clearly demonstrate the importance of utilizing moving-water well-balanced methods for solutions near a moving-water equilibrium. There have been a few attempts in developing well-balanced methods for the moving water equilibrium. In [14], Gosse developed a class of first order accurate flux-vector-splitting schemes based on the theory of non-conservative products, which is well-balanced for general steady states including moving water equilibria. Jin and Wen proposed the interface type method [16–18,33] to capture the general equilibria with second order accuracy. Russo [27] developed well-balanced central schemes on staggered grids which are second order accurate. Bouchut and Morales [4] proposed numerical methods based on local subsonic steady state reconstruction, which are exactly well-balanced for subsonic moving equilibria. A few high order accurate well-balanced methods for the moving water equilibrium have been introduced recently. In [22], well-balanced finite volume weighted essentially non-oscillatory (WENO) methods are designed for arbitrary equilibria of the shallow water equations. The key component there is a special way to recover the moving water equilibrium and a well-balanced quadrature rule of the source term. Russo and Khe [28] presented well-balanced central WENO schemes, and recently, Castro et al. [7] proposed a new well-balanced finite volume WENO scheme in the framework of path-conservative methods.

In recent years, high order accurate numerical schemes, including finite difference/volume WENO schemes, spectral methods and discontinuous Galerkin (DG) methods, have been developed to reduce the number of computational cells and minimize the computational time to achieve the desired resolution. Among these methods, DG method is a class of finite element methods using discontinuous piecewise polynomial space as the solution and test function spaces (see [8] for a historic review). It combines advantages of both finite element and finite volume methods, and has been successfully applied to a wide range of applications. Several advantages of the DG method, including its accuracy, high parallel efficiency, flexibility for hp-adaptivity and arbitrary geometry and meshes, make it particularly suited for the shallow water equations [12,11,13,36].

However, high order moving-water well-balanced WENO methods developed in [22] and [7] cannot be generalized to DG methods directly. The first difficulty encountered comes from the fact that the solution function space in DG methods is the piecewise polynomial space, while the water height h in a moving water equilibrium is in general not a polynomial. Other difficulty includes that the approximation of the source term now becomes the integral of the source term multiplied by the test function, therefore the quadrature rule idea in [22] cannot be applied directly. The main objective of this paper is to develop positivity-preserving high order accurate well-balanced DG methods for the shallow water equations with moving water equilibrium. This will be the first paper to achieve this goal, to our best knowledge. We first present the transformation between the conservative variables and equilibrium variables following the techniques presented in [22]. The recovery of well-balanced states is obtained through the computation of the reference states. The numerical solution is then decomposed into the equilibrium part and the remaining part. Then, a special, well-balanced source term approximation can be derived based on this decomposition and we will show it balances the effect of numerical fluxes which are computed by a generalized hydrostatic reconstruction. All together this leads to a rather transparent formulation of our well-balanced DG scheme. Another important difficulty often encountered in the simulations of the shallow water equations is the appearance